1 CHAPTER 1 Real Numbers

WHAT THIS CHAPTER BUILDS
You will learn to break any number into primes and use that one idea to find HCF and LCM, and to prove that root 2 can never be written as a fraction.

NUMBER SYSTEMS UNIT

IN THIS CHAPTER

Kabir and Priya pack mangoes into equal trays, one tray full, one part filled, one still empty.

Getting started

Primes decide everything

A STORY

The heap and the trays

Kabir and Priya pack mangoes at the school stall. The heap holds 60 mangoes. Every tray must take the same number, with none left over.

They use trays of 12. One tray stands full already. A second tray is only part filled. A third tray is still empty.

Priya checks the full tray against her notes. “Twelve fits into sixty five times, exactly,” she says.

“Could a tray of 15 work too?” Kabir asks, lifting another mango from the heap. “Or would some mangoes be left over?”

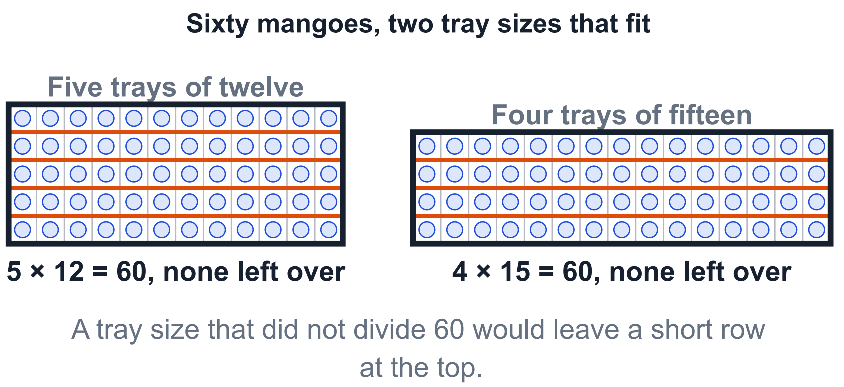
Priya works it out. “60 divided by 15 is 4,” she says. “That divides evenly too.”

“So two different tray sizes both divide the heap exactly,” Kabir says. “How many sizes can do that?”

Priya has no answer yet. Neither do you. How many tray sizes split sixty exactly, with nothing left over?

RULE
When a question asks what a number can or cannot be, break it into primes first. Then read the answer off the primes.

KEY
Every whole number bigger than 1 breaks into primes one way only. $60 = 2 \cdot 2 \cdot 3 \cdot 5$, and nothing else.



Kabir and Priya’s sixty mangoes are packed twelve to a tray in five trays, and fifteen to a tray in four trays, none left over.

Let us break 60 into smaller pieces. Split it once, then break each piece again, until only primes are left.

$$60 = 4 \cdot 15$$

$$4 = 2 \cdot 2$$

$$15 = 3 \cdot 5$$

$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$

Now start a different way. Pull out a 2 first, and keep going.

$$60 = 2 \cdot 30$$

$$30 = 2 \cdot 15$$

$$15 = 3 \cdot 5$$

$60 = 2 \cdot 2 \cdot 3 \cdot 5$

The last line is the same as before.

We started two different ways. We reached the same four primes: 2, 2, 3 and 5. Try a third way yourself, starting with $60 = 6 \cdot 10$. You will reach the same four primes again.

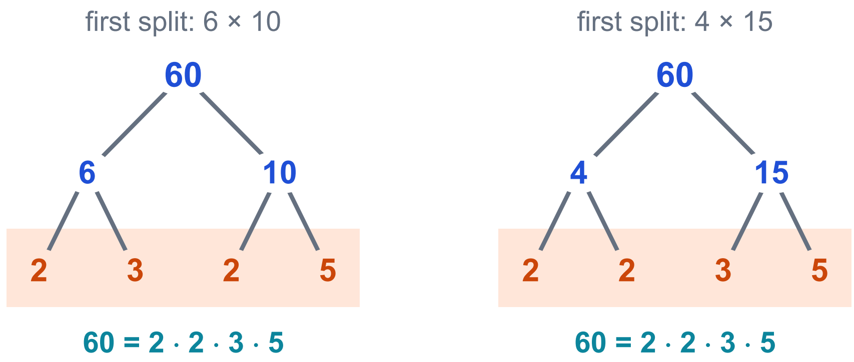
This always happens. Every whole number bigger than 1 breaks into primes in one way only. You can write the primes in a different order. You cannot get different primes. This fact has a name: the Fundamental Theorem of Arithmetic.

In this chapter we use this one fact for two jobs.

- **Job one:** finding the HCF and LCM of two numbers from their primes.
- **Job two:** proving that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ can never be written as a fraction.

Both jobs rest on the same fact. A number has one prime factorisation, and no other.

Two ways to split 60, the same four primes



Both trees start at 60 and split it differently, and both finish on the same four primes, 2, 2, 3 and 5.

A SCHOLAR INDIA REMEMBERS

Kanada lived in north India more than two thousand years ago. He taught that all matter is built from tiny parts that cannot be split. Two parts join, or three parts join, and bigger things are made. Let us do the same with a number. We split 60 until no piece splits again: $60 = 2 \times 2 \times 3 \times 5$. *Those last pieces are primes.*

Kanada · the sparrow

IN THE WILD

A shopkeeper has 60 mangoes to pack. Every box must hold the same number, with none left over. So only a box size that divides 60 will work. Every one of those sizes comes out of $60 = 2 \cdot 2 \cdot 3 \cdot 5$. Split the pile any way you like. You land on those same four primes every time. That is why the HCF and the LCM of two numbers can be read from their primes.

CHECK YOURSELF

1. **The Fundamental Theorem of Arithmetic states that every integer greater than** 1

A. factors into primes in exactly one way, including a fixed order

B. can be factored into primes in more than one way if the number is large enough

C. factors into primes in exactly one way, apart from the order of the factors

D. is itself always a prime number

2. This chapter uses the Fundamental Theorem of Arithmetic to prove that $\sqrt{2}$ cannot be written as a fraction. Which property of the theorem does that proof actually need?

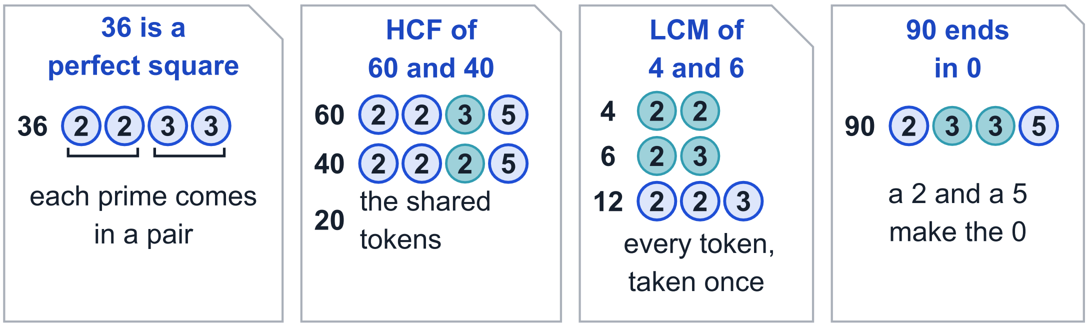
- **A.** that every number has at least one prime factor
- **B.** that prime numbers are infinite in number
- **C.** that a number's prime factorisation is fixed
- **D.** that composite numbers can always be factored eventually

1 more in this set online.

IN EVERY CHAPTER

Before you start

Four checks, one picture: primes as tokens



Every check on this page is a question about tokens: which ones pair up, which are shared, which appear at all. The chapter never needs more than that.

Four panels write the primes of 36, of 60 and 40, of 4 and 6, and of 90 as tokens to answer each check.

DO THIS FIRST

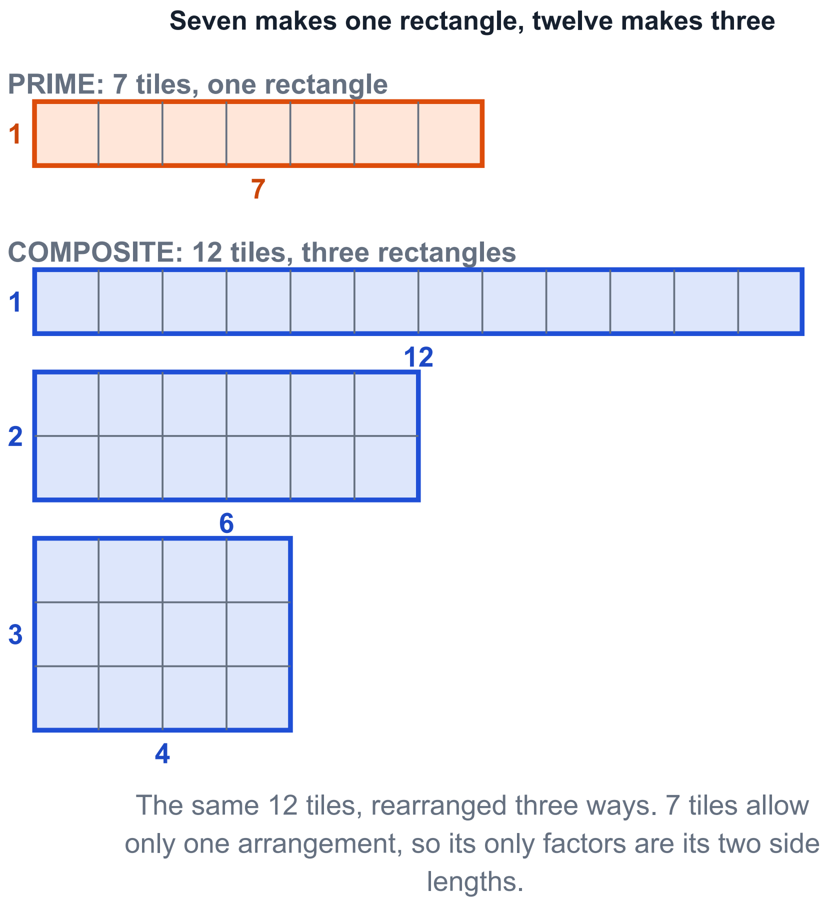
You already know what a prime number is. Now numbers break apart in a new way. Try each check below. It takes a minute.

- **A prime number's factors** (Class 8, Ganita Prakash Part 1, page 3).
  Here: every factorisation, like $60 = 2^2 \cdot 3 \cdot 5$, uses only primes.
  Check: 7 has only two factors, 1 and 7.
- **A number's factorisation** (Class 8, Ganita Prakash Part 1, page 9).
  Here: every number breaks into primes in only one way, in any order.
  Check: $36 = 2^2 \cdot 3^2$ is a perfect square, since each power is even.
- **The HCF of two numbers** (Class 8, Ganita Prakash Part 1, page 161).
  Here: the HCF keeps the smallest power of each shared prime.
  Check: 60 and 40 have HCF 20.
- **The LCM of two numbers** (Class 8, Ganita Prakash Part 1, page 121).
  Here: the same LCM comes from two numbers' largest prime powers.
  Check: 4 and 6 have LCM 12.
- **Lowest terms of a fraction** (Class 9, Ganita Manjari, page 58).
  Here: the root 2 proof assumes the fraction is written this same way.
  Check: $3/20$ has no common factor other than 1.

- **A number divisible by 10** (Class 8, Ganita Prakash Part 1, page 123).
  Here: a number ending in zero has both 2 and 5 as factors.
  Check: 90 ends in 0, and $90 = 2 \cdot 3^2 \cdot 5$.

If any of these felt new, read the page named before going on.

Prime and composite numbers



Seven tiles can be pushed into one rectangle and no other, so seven has only two factors. Twelve tiles make three, so twelve has six.

Take the number 7. Which numbers divide 7 exactly, with no remainder left over? Try 2: $7 \div 2 = 3$ remainder 1, so 2 does not work. You try 3, 4, 5 and 6 the same way. None of them work either. Only 1 and 7 divide 7 exactly. So 7 has exactly two factors: 1 and 7. A number with exactly two factors is called PRIME.

Now take 12. Which numbers divide 12 exactly? 1, 2, 3, 4, 6 and 12 all do. That is six factors, more than two. A number with more than two factors is called COMPOSITE. Every composite number can be built by multiplying primes together. $12 = 2 \cdot 2 \cdot 3$. We also write this as $12 = 2^2 \cdot 3$. The small 2 in 2^2 means 2 is used twice.

What about 1? Only 1 divides 1 exactly. That is one factor, not two. So 1 is not prime. It is not composite either, since it does not have more than two factors. 1 is the one number that is neither.

When we write a number using only primes multiplied together, we call it a PRIME FACTORISATION. $60 = 2^2 \cdot 3 \cdot 5$ is a prime factorisation of 60. But $60 = 4 \cdot 15$ is not a prime factorisation, because 4 and 15 are not primes. $4 = 2 \cdot 2$, and $15 = 3 \cdot 5$, so both break apart further.

Your turn: is 15 prime or composite? Find all its factors first. (Answer: 1, 3, 5 and 15. Four factors, so 15 is composite.)

TRY
How many factors does a prime number have?

Answer: Two. They are 1 and the number itself.

BACK
Primes decide everything · p. 3

RULE
1 is neither prime nor composite. Do not count it as either.

A SCHOLAR INDIA REMEMBERS

What kind of number is 1, really? Let us check. A prime has exactly two factors. 7 has two factors, 1 and 7, so 7 is prime. A composite number has more than two. 1 has only one factor, itself. So 1 is not prime, and it is not composite. It is a kind of its own.

Kanada · the sparrow

CHECK YOURSELF

3. A composite number is a positive integer that

- **A.** cannot be divided evenly by any smaller number
- **B.** has more than two positive factors
- **C.** is always an even number
- **D.** has exactly two positive factors

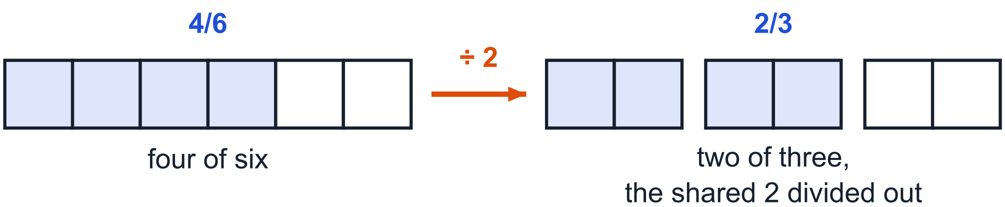
4. A student claims: “the number 1 is the smallest prime number, since it only divides itself.” What is wrong with this claim?

- **A.** 1 is actually composite, not prime
- **B.** 1 has only one factor
- **C.** the claim is correct — 1 is indeed the smallest prime
- **D.** primes must be even, so 1 cannot be prime

2 more in this set online.

A rational number in lowest terms

Four of six is two of three



The shaded amount never changes. Dividing out the shared factor only changes how the tiles are grouped, and once no shared factor is left the fraction is in lowest terms.

The same four shaded tiles read as four of six, then divide by 2 to read as two of three.

Take $3/4$. It is a fraction of two integers, so it is RATIONAL. Now take 5. Write it as $5/1$ yourself, and check that it fits the same pattern. Any number we can write as p/q, for integers p and q with $q \neq 0$, is rational.

Now take $4/6$. Let us simplify it. Both 4 and 6 share the factor 2. We divide it out: $4/6 = 2/3$. $2/3$ has no factor left to divide out, so it is in LOWEST TERMS. Its numerator and denominator are then COPRIME. **A fraction is in lowest terms when its numerator and denominator share no factor except 1. $2/3$ is in lowest terms; $4/6$ is not.**

A number that cannot be written as p/q at all, in any form, is IRRATIONAL.

Two more facts about rationals. Add, subtract or multiply two rationals and the answer is rational again. Divide them and it is rational again too, with one catch: you may never divide by 0. So take a rational r that is not 0. Then $1/r$ is rational — turn p/q upside down and you get q/p, a fraction of two integers again. And s/r is rational for any rational s, since it is s times $1/r$.

When we meet a rational number in a proof, we assume it is already written in lowest terms. This is not a small detail to skip past. It is what makes a contradiction proof work. Without the assumption, there is nothing left for the contradiction to catch.

Your turn: is $6/9$ in lowest terms? (Answer: No. 3 divides both 6 and 9. In lowest terms it is $2/3$.)

TRY
Is $4/6$ in lowest terms?

Answer: No. 2 divides both. In lowest terms it is $2/3$.

RULE
Start with a fraction in lowest terms. Then the top and the bottom share no factor, and a shared factor later is impossible.

KEY
In lowest terms the top and the bottom share no factor except 1. $2/3$ is in lowest terms. $4/6$ is not.

One point, two names

Both names sit at the same point, so $4/6$ and $2/3$ are the same number.

CHECK YOURSELF

5. A rational number, by definition, can be written as

- **A.** p/q for integers p and q with $q \neq 0$
- **B.** p/q for integers p and q with $p \neq 0$
- **C.** $\sqrt{p}$ for any integer p
- **D.** a terminating decimal only

6. A fraction p/q is in lowest terms when

- **A.** p and q are both prime numbers
- **B.** q is as small as possible, regardless of p
- **C.** p and q share no factor but 1
- **D.** p is greater than q

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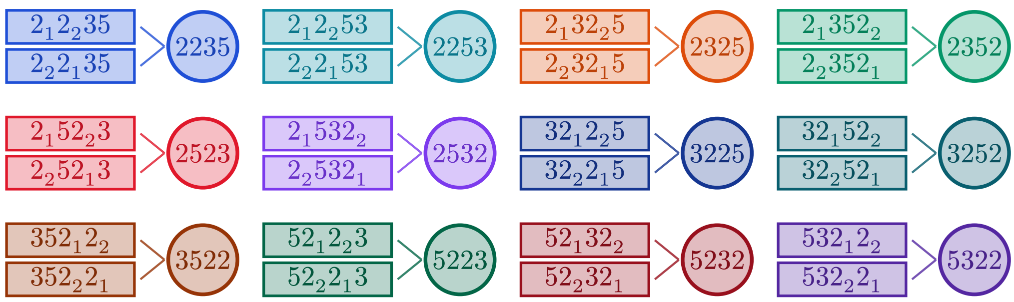
The ideas

Every composite number factors into primes, uniquely

24 extraction orders, 12 that a reader can tell apart



24 tagged extraction orders (left, in pairs) collapse onto 12 visible sequences (right, one circle each). Each pair differs only by swapping 2_1 and 2_2 – a swap that changes nothing a reader can see, which is why there are exactly two chips per circle.

Sixty's four primes are 2, 2, 3 and 5, and the 24 arrangements are four choices, then three, then two, then one.

Let us state the Fundamental Theorem of Arithmetic plainly. Every composite number breaks into primes in exactly one way. Write the primes in a different order, and it is still the same set of primes. No second, different set exists.

Try it on 1400. Factor it one way: $1400 = 2^3 \cdot 5^2 \cdot 7$. Now factor it a different way. Pull out a 5 first instead of a 2. You still land at $2^3 \cdot 5^2 \cdot 7$. No other set of primes multiplies out to 1400.

This says more than just that primes exist for every number. It says the primes are fixed. Break 1400 apart any way you like. The primes that appear, and how many times each one appears, come out the same every time.

Your turn: could 84 have two different prime factorisations? (Answer: No. $84 = 2^2 \cdot 3 \cdot 7$ is its only prime factorisation.)

BACK
Prime and composite numbers · p. 6

TRY
Could 1400 have a second, different prime factorisation?

Answer: No. Every whole number has exactly one.

RULE
Keep dividing by the smallest prime that goes in. Stop when 1 is left. For 60 that is 30, then 15, then 5, then 1.

A SCHOLAR INDIA REMEMBERS

Will the smallest parts come out the same every time? Test a new number. Split 84 *as* 4×21. *That gives* $2 \times 2 \times 3 \times 7$. *Now split* 84 *as* 2×42. *That gives* $2 \times 2 \times 3 \times 7$ *again. Two different starts, and the same four primes.*

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CHECK YOURSELF

7. The Fundamental Theorem of Arithmetic says every composite number factors into primes

- **A.** in exactly one way, apart from order
- **B.** in exactly one way, including a fixed order
- **C.** in more than one way for some large numbers
- **D.** only if the number is even

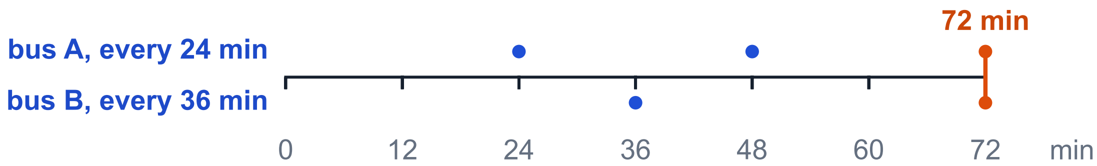
8. $1400 = 2^3 \cdot 5^2 \cdot 7$. **According to the Fundamental Theorem of Arithmetic, what can be said about any OTHER prime factorisation of** 1400**?**

- **A.** no other set of primes multiplies to 1400
- **B.** a second factorisation exists using different primes, but it is rarely found
- **C.** a second factorisation exists only if 1400 is written in a different number base
- **D.** the theorem does not apply to numbers with three distinct prime factors

2 more in this set online.

Reading HCF and LCM off two factorisations

Two rows of stops, one shared tick



The same reading works for any two repeating times a problem gives you, not only 24 and 36 minutes.

HCF sits below, LCM sits above – for two numbers or three

The HCF sits at or below every number, and the LCM sits at or above every number, for both groups shown.

Take 8 and 12. First write each one as primes.

$$8 = 2 \cdot 2 \cdot 2$$

$$12 = 2 \cdot 2 \cdot 3$$

Now let us find the HCF. The HCF is the biggest number that divides both 8 and 12. Look at the prime 2. It appears three times in 8 and two times in 12. Take the smaller count, two. The prime 3 is only in 12, not in 8, so it is not shared. Leave it out. So $\text{HCF} = 2 \cdot 2 = 4$. Check: 4 divides 8. 4 divides 12. Yes.

Now the LCM. The LCM is the smallest number that both 8 and 12 divide into. This time take every prime that appears in either list, and take the bigger count. The prime 2: three times in 8, two times in 12. Take three. The prime 3: once, in 12. Take one. So $\text{LCM} = 2 \cdot 2 \cdot 2 \cdot 3 = 24$. Check: $24 \div 8 = 3$, and $24 \div 12 = 2$. Both exact.

What if the two numbers share no prime at all? Take $8 = 2 \cdot 2 \cdot 2$ and $15 = 3 \cdot 5$. Go looking for a prime on both lists and there is none. So there is nothing to multiply together, and the HCF is 1: 1 divides every number, and here nothing bigger divides both. Two numbers whose HCF is 1 are COPRIME. The LCM is read off exactly as before, from every prime in either list: $\text{LCM}(8, 15) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$.

You can use the same method for three numbers or more. Nothing changes. It also answers a question like this: one bell rings every 8 minutes, and another every 12 minutes. When do they ring together again? The answer is the LCM, 24 minutes.

Rule: the HCF uses the smaller count of each prime the numbers share. The LCM uses the bigger count of every prime that appears. For 8 and 12, $\text{HCF} = 4$ and $\text{LCM} = 24$.

Your turn: a prime appears in only one of two numbers. Does it appear in their HCF? (Answer: No. The HCF only uses primes that both numbers share.)

BACK
Every composite number factors into primes, uniquely · p. 9

TRY
A prime appears in only one of the two numbers. Does it appear in their HCF?

Answer: No. The HCF uses only primes that both numbers share.

RULE
Break both numbers into primes first. Then compare the powers of one prime at a time.

A SCHOLAR INDIA REMEMBERS

Before you look for the HCF, make a clean list of parts. $12 = 2 \times 2 \times 3$. $18 = 2 \times 3 \times 3$. *Which primes are on both lists? One* 2 *and one* 3. *So the HCF is* $2 \times 3 = 6$. *Now tell me: which primes will the LCM need?*

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CHECK YOURSELF

9. To find the HCF of two numbers from their prime factorisations, take

- **A.** the largest power of every prime the two numbers share
- **B.** the smallest power of every prime the two numbers share
- **C.** the smallest power of every prime appearing in either number
- **D.** the product of all shared primes, ignoring their powers

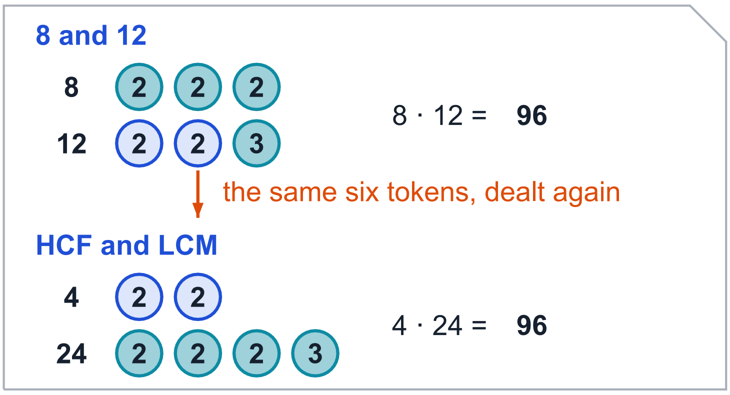
10. To find the LCM of two numbers from their prime factorisations, take

- **A.** the largest power of every prime appearing in either number
- **B.** the largest power of every prime the two numbers share
- **C.** the smallest power of every prime appearing in either number
- **D.** the sum of all powers of every shared prime

2 more in this set online.

HCF times LCM equals the product

Six tokens, dealt two ways



8 and 12 together hold six prime tokens. The HCF takes the shared ones and the LCM takes the rest, so the two products are the same six tokens multiplied: 96 both ways.

The prime tokens of 8 and 12 split into the shared tokens for the HCF, 4, and the rest for the LCM, 24.

Take 8 and 12 again. Their HCF is 4. Their LCM is 24. Let us multiply the HCF by the LCM, then multiply the two original numbers.

$$4 \cdot 24 = 96$$

$$8 \cdot 12 = 96$$

The two answers match. This is not a coincidence for this one pair.

For any two positive integers a and b: $\mathrm{HCF}(a, b) \cdot \mathrm{LCM}(a, b) = a \cdot b$. Once you know any three of HCF, LCM, a and b, this relation finds the fourth. You only need to divide. No fresh factorising needed.

Here it is, run backwards. The HCF of 24 and 36 is 12. What is their LCM? Put the three numbers you know into the relation, then divide.

$$12 \cdot \mathrm{LCM} = 24 \cdot 36 = 864$$

$$\mathrm{LCM} = 864 \div 12 = 72$$

Check it the long way if you like: $24 = 2^3 \cdot 3$ and $36 = 2^2 \cdot 3^2$, so the LCM is $2^3 \cdot 3^2 = 72$. The same answer, and the short way never factorised anything.

The relation works for a PAIR of numbers only. For three integers, the product of all three is usually NOT equal to the product of their HCF and LCM. Check a third number by prime factorisation directly. Do not carry this two-number shortcut over.

Your turn: for 4 and 6, the HCF is 2 and the LCM is 12. Does HCF $\cdot$ LCM equal $4 \cdot 6$? (Answer: Yes. $2 \cdot 12 = 24$, and $4 \cdot 6 = 24$.)

CAUTION
Do not use this on three numbers. Break them into primes instead.

RULE
$\mathrm{HCF} \cdot \mathrm{LCM} = a \cdot b$. Know three of the four, and divide to get the fourth.

KEY
This works for two numbers only. It breaks down for three.

CHECK YOURSELF

11. For any two positive integers a and b, $\mathrm{HCF}(a,b) \cdot \mathrm{LCM}(a,b)$ equals

- **A.** $a^2 \cdot b^2$
- **B.** $a \cdot b$
- **C.** $a + b$
- **D.** $a - b$

12. If $\mathrm{HCF}(a,b) = 4$ and $a \cdot b = 96$, the relation $\mathrm{HCF} \cdot \mathrm{LCM} = a \cdot b$ gives

- **A.** $\mathrm{LCM}(a,b) = 384$, found by multiplying 96 by 4
- **B.** $\mathrm{LCM}(a,b) = 92$, found by subtracting 4 from 96
- **C.** $\mathrm{LCM}(a,b) = 24$, found by dividing 96 by 4
- **D.** $\mathrm{LCM}(a,b) = 4$, the same as the HCF

2 more in this set online.

A prime dividing a square divides the root

The proof that root 2 is irrational reaches the line a squared is 2 b squared, and the condition at the top comes from there.

Here is a fact we will use many times. If p is a prime number and p divides a^2, then p also divides a, for any integer a. We will not prove this fact. We will use it as a tool.

Check it on a small case. Take $a = 6$, so $a^2 = 36$. The prime 2 divides 36. 2 also divides 6. Now try $a = 10$ yourself: $a^2 = 100$. The prime 5 divides 100. Does 5 divide 10 too? Yes. You just checked two examples, and both confirm the fact.

Why is this true? It comes from the Fundamental Theorem of Arithmetic. A number's prime factorisation never changes, no matter how it is written. The primes in a^2's factorisation are exactly the primes already in a's own factorisation, each one appearing twice. So if p shows up in a^2, p was already there in a.

We use this one fact in every proof that a square root is irrational: $p = 2$ for $\sqrt{2}$, $p = 3$ for $\sqrt{3}$, $p = 5$ for $\sqrt{5}$.

RULE
Use $p = 2$ for $\sqrt{2}$. Use $p = 3$ for $\sqrt{3}$, and $p = 5$ for $\sqrt{5}$.

KEY
If a prime p divides a^2, then p divides a. 2 divides 36, and 2 divides 6.

CHECK YOURSELF

13. If p is prime and p divides a^2, the key fact used throughout this chapter's proofs says

- **A.** p must also divide a itself
- **B.** p must divide a twice, since squares double every factor
- **C.** p divides a^2 only when a is itself prime
- **D.** p divides a only when p is even

14. This chapter states the "prime divides a square" fact WITHOUT proving it, but says it follows from

- **A.** a separate rule about even numbers, unrelated to prime factorisation
- **B.** the definition of a rational number in lowest terms
- **C.** a property that holds only for the specific primes 2, 3 and 5
- **D.** the Fundamental Theorem of Arithmetic's uniqueness

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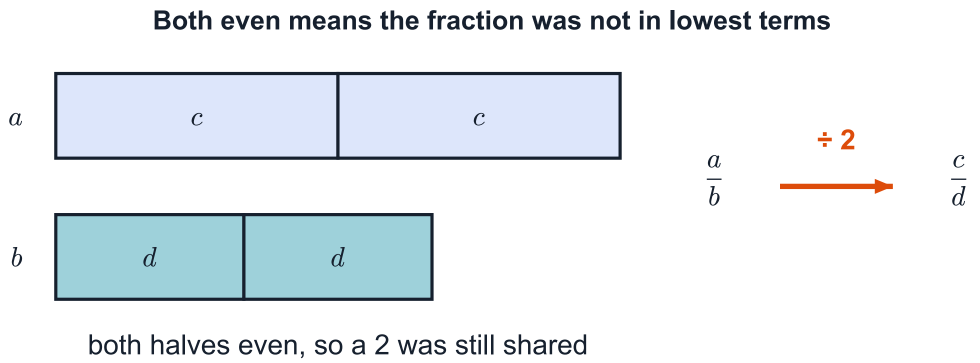
Root 2 cannot be written as a fraction



The proof assumed a/b had no shared factor. It then forced 2 into both a and b. A fraction that can still be halved top and bottom was never in lowest terms, so the assumption was false.

The fraction a over b, with both a and b even, halves again to c over d.

We have built three tools already.

- **Prime factorisation.** Every whole number bigger than 1 breaks into primes.
- **One factorisation only.** A number's prime factorisation never changes.
- **The lemma.** A prime dividing a square must divide the number itself.

Let us put all three together. They prove something concrete.

THEOREM. $\sqrt{2}$ is irrational.

Here is the plan, before you read the numbered proof. Assume $\sqrt{2}$ is a fraction. Follow where that leads. Watch it collide with something impossible. That collision is the whole proof.

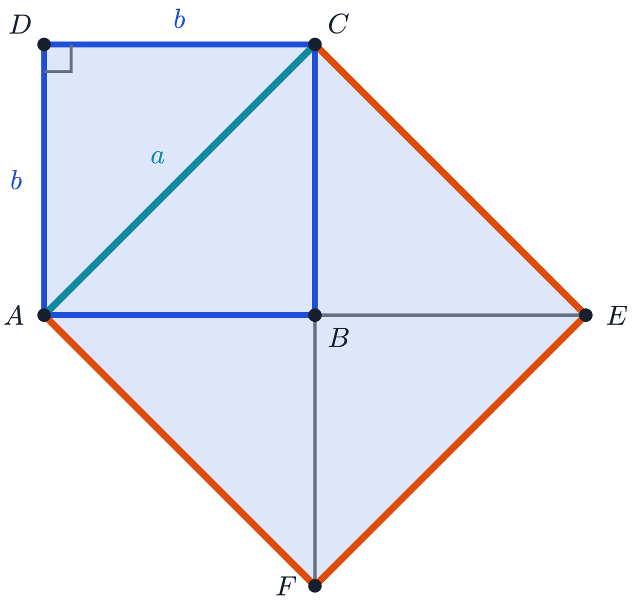
We call this proof by contradiction. You assume the opposite of what you want to prove. Then you show that assumption forces something impossible. The numbered proof below gives every step its own reason. Do not skip a step. Each one is a gap the proof cannot survive without.

BACK
A rational number in lowest terms · p. 7

CAUTION
Do not stop once 2 divides a. Run the same argument again on b. The contradiction shows up only then.

RULE
Every step needs a reason. Dividing out the shared factor matters as much as the impossible line at the end.

The square on the diagonal is two b-squares



The square on diagonal a splits into four triangles, and the square of side b splits into two, giving a squared equals 2 b squared.

Given: A rational number written in lowest terms is p/q. Here p and q are integers. They share no common factor except 1. We also have the lemma from before. If a prime divides a^2, it divides a too.

To Prove: $\sqrt{2}$ is irrational.

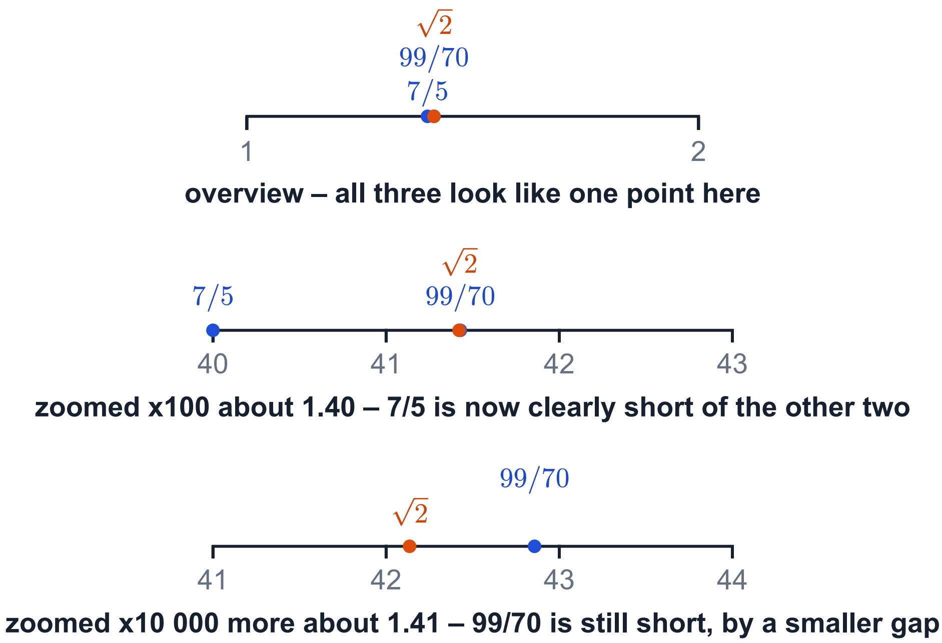
Proof:

	Statement	Reason
1.	assume, to the contrary, that $\sqrt{2}$ is rational	*We assume this. The rest of the proof will show it cannot be true.*
2.	$\sqrt{2} = a/b$ for integers $a, b \neq 0$, with a and b coprime	*Every rational number can be written this way, in lowest terms.*
3.	$2 = a^2/b^2$, so $a^2 = 2b^2$	*Square both sides of $\sqrt{2} = a/b$. Then multiply both sides by b^2.*
4.	2 divides a^2	*$a^2 = 2b^2$. This says a^2 is 2 times a whole number. So 2 divides a^2.*
5.	2 divides a	*2 is prime, and 2 divides a^2. By the lemma, 2 must divide a itself.*
6.	$a = 2c$ for some integer c	*If 2 divides a, then a is 2 times some whole number. Call that number c.*
7.	$(2c)^2 = 2b^2$, so $b^2 = 2c^2$	*Put $a = 2c$ into $a^2 = 2b^2$. $(2c)^2 = 4c^2$, so $4c^2 = 2b^2$. Divide both sides by 2. $b^2 = 2c^2$.*

8.	2 divides b^2	*$b^2 = 2c^2$. This says b^2 is 2 times a whole number. So 2 divides b^2.*
9.	2 divides b	*This is the same lemma again. 2 is prime, and 2 divides b^2. So 2 divides b.*
10.	2 divides both a and b	*But a and b were assumed coprime. They share no factor but 1. **This is the contradiction the whole proof was built to reach.***
11.	$\sqrt{2}$ is irrational	*The assumption that $\sqrt{2}$ is rational led to something impossible. So the assumption is false. $\sqrt{2}$ must be irrational.*

∎

Closing in on $\sqrt{2}$, and never landing



No fraction, no matter how large its denominator, ever closes this gap, which is the endless search one proof by contradiction replaces.

CHECK YOURSELF

15. Step (2) of the $\sqrt{2}$ proof writes $\sqrt{2} = a/b$ with a and b coprime. Why must this fraction be assumed in LOWEST TERMS?

- **A.** to make the arithmetic in step (3) simpler
- **B.** because $\sqrt{2}$ is only rational when written in lowest terms
- **C.** so that the later contradiction
- **D.** lowest terms is not required; any a/b would work equally well

16. Suppose the proof skipped the coprime assumption in step (2) and just wrote $\sqrt{2} = a/b$ for any integers a, b. Which step stops working?

- **A.** step (10), the contradiction that needs coprime a, b
- **B.** step (3), where squaring both sides would no longer be valid
- **C.** step (5), where the prime-divides-square lemma would no longer apply
- **D.** no step fails — the proof reaches the same contradiction either way

3 more in this set online.

Common mistakes

HCF times LCM works for two numbers only

FIRST TRY

~~HCF of 9, 15 and 21 is 3, so I wrote~~

~~LCM $= (9 \cdot 15 \cdot 21)/3 = 2835/3 = 945$.~~

SECOND LOOK

That rule only holds for a pair. Back to primes, $9 = 3^2$, $15 = 3 \cdot 5$, $21 = 3 \cdot 7$, so LCM $= 3^2 \cdot 5 \cdot 7 = 315$.

HCF times LCM equals the product only for a pair. For three numbers or more, build the LCM from the primes.

DOES HCF TIMES LCM EQUAL THE PRODUCT, FOR THREE NUMBERS TOO?

WEAK

For 9, 15 and 21: the HCF is 3. Carrying the two-number rule across, LCM $= (9 \cdot 15 \cdot 21)/3 = 2835/3 = 945$.

That feels right — it reused a trusted shortcut. But check it: $945/3 = 315$ is still a common multiple of 9, 15 and 21. If 945 divides by 3 and still works, 945 was never the LEAST common multiple.

STRONG

The relation HCF $\cdot$ LCM $= a \cdot b$ is proved for a PAIR only. For three numbers, go back to the primes: $9 = 3^2$, $15 = 3 \cdot 5$, $21 = 3 \cdot 7$. Take the largest power of each prime: LCM $= 3^2 \cdot 5 \cdot 7 = 315$.

Three numbers can share a prime in pairs without all three sharing it — 9 and 21 share an extra factor of 3 that 15 does not — and the product rule has no way to count that. 315, not 945, is the true LCM.

BACK

HCF times LCM equals the product · p. 12

CAUTION

Two numbers only. For three, go back to the primes and take the largest power of each.

KEY

Test the rule on 2, 3 and 4. It claims LCM $= 24$, but 12 is already a multiple of all three.

Checking a few fractions is not a proof

Let us try a shortcut. Guess a fraction for $\sqrt{2}$. Check a few candidates, like $7/5$ or $99/70$. Stop once none of them work.

That is not a proof. Infinitely many fractions are still untried. Checking a finite number of them can never rule out the rest. A fraction that happens to equal $\sqrt{2}$ could be sitting further down the list than you looked.

Only a proof by contradiction actually settles the question. Assume a lowest-terms fraction equals $\sqrt{2}$. Then show that assumption is impossible, the way the proof above does.

Checking more fractions only makes you more sure. It never rules out the fractions still untried.

Your turn: you check 10 fractions and none equals $\sqrt{2}$. Have you proved $\sqrt{2}$ is irrational? (Answer: No. Infinitely many fractions are still untried.)

DOES CHECKING A FEW FRACTIONS EVER PROVE THE SQUARE ROOT OF 2 IS IRRATIONAL?

WEAK

A student wants to know if $\sqrt{2}$ is irrational. They try $7/5$. They square it: $(7/5)^2 = 49/25 = 1.96$. Close to 2, but not 2. They try $99/70$. $(99/70)^2 = 9801/4900 \approx 2.0002$. Even closer, but still not 2. They try eight more fractions. None of them works. "I have checked ten fractions," the student says. "None of them equals $\sqrt{2}$. So $\sqrt{2}$ must be irrational."

This feels like proof. It is not. Infinitely many fractions are still untried. Checking ten fractions, or a thousand, never rules out the rest. The next one you try could be the one that works.

STRONG

We assume, to the contrary, that some fraction a/b, in lowest terms, equals $\sqrt{2}$. Let us follow where that leads. Squaring gives $a^2 = 2b^2$. So 2 divides a^2. So 2 divides a. The same steps then force 2 to divide b too. But a and b were coprime. They cannot both be divisible by 2. This is impossible.

The assumption is what breaks, not the arithmetic. Since assuming a fraction equals $\sqrt{2}$ leads to something impossible, no fraction can equal $\sqrt{2}$. This checks every fraction at once, not one at a time.

CAUTION

Not finding a fraction is not the same as showing there is none.

KEY

You can test a thousand fractions and learn nothing. Infinitely many are left.

WATCH OUT

The trap: The student tries five or six fractions, finds none equal to $\sqrt{2}$, and stops, calling the search finished.

The correction: Each fraction you try rules out exactly one of infinitely many, so a hundred tries leave just as many untried as before.

CHECK YOURSELF

17. A student checks 1.4^2, 1.41^2 and 1.414^2, finds none exactly equal 2, and concludes $\sqrt{2}$ is irrational. Is this a valid proof?

- **A.** yes — checking three cases is enough to establish a mathematical fact
- **B.** yes, as long as the numbers checked are close enough to $\sqrt{2}$
- **C.** no — finitely many checks cannot rule out infinitely many fractions
- **D.** no, but only because the student did not check enough decimal places

18. Which of these WOULD count as a valid proof that a number x is irrational?

- **A.** computing x to 50 decimal places and finding no repeating pattern
- **B.** checking every fraction with denominator under 1000 and finding none equal x
- **C.** assuming $x = a/b$ in lowest terms and deriving a contradiction
- **D.** asking whether x "looks like" a nice number

1 more in this set online.

Worked examples

WORKED EXAMPLE · Factorising 378 by repeated division

1. $378/2 = 189$

Start with the smallest prime that goes in. 378 is even, so that prime is 2.

2. $189/3 = 63$, $63/3 = 21$, $21/3 = 7$

189 is odd, so no second 2 comes out. Move up to the next prime, 3, and keep dividing while it still goes in. Here it goes in three times.

3. $7/7 = 1$

7 is prime, so it divides itself. The quotient is 1 and the chain stops.

4. $378 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 7 = 2 \cdot 3^3 \cdot 7$

Collect the divisors you used, in the order you used them. The three 3s are written 3^3.

5. $3^3 = 27$, $2 \cdot 27 \cdot 7 = 378$

CHECK ***Multiply the primes back yourself.*** *You started at 378, so you must land on 378.*

TRY
Find the prime factorisation of 132.

Answer: $132/2 = 66$, $66/2 = 33$, $33/3 = 11$, $11/11 = 1$, so $132 = 2^2 \cdot 3 \cdot 11$.

CAUTION
Stopping at a quotient that is not itself prime leaves the job half done. $378 = 2 \cdot 189$ is not a prime factorisation.

RULE
Always divide by the SMALLEST prime that still goes in. Stop when the quotient is 1.

FIND THE PRIME FACTORISATION OF ANY NUMBER, BY REPEATED DIVISION.

1. **Divide by the smallest prime** For 378, that prime is 2, because 378 is even. $378/2 = 189$.
2. **Keep going with that prime** 189 is odd, so 2 is finished. The next prime to try is 3. $189/3 = 63$, $63/3 = 21$, $21/3 = 7$. Now 3 is finished too.
3. **Stop at a quotient of one** Now 7 is prime, so it divides itself. $7/7 = 1$, and there is nothing left to divide.
4. **Collect the divisors, then check** The divisors used were $2, 3, 3, 3, 7$, so $378 = 2 \cdot 3^3 \cdot 7$. Multiply them back to check. $3^3 = 27$ and $2 \cdot 27 \cdot 7 = 378$. It matches, so the factorisation is right.

WORKED EXAMPLE · HCF and LCM of 12 and 18

1. $12 = 2^2 \cdot 3$, $18 = 2 \cdot 3^2$

Factor each number into primes first. Then you can compare them, prime by prime.

2. $\text{HCF} = 2^1 \cdot 3^1 = 6$

Take the smaller power of every prime the two numbers share. Here that means 2^1 against 2^2, and 3^1 against 3^2.

3. $\text{LCM} = 2^2 \cdot 3^2 = 36$

Take the larger power of every prime that appears in either number.

4. $6 \cdot 36 = 216$, $12 \cdot 18 = 216$

CHECK ***Check the product relation yourself.*** *Multiply your HCF by your LCM. Then multiply the two original numbers. Both give 216.*

BACK
Reading HCF and LCM off two factorisations · p. 10

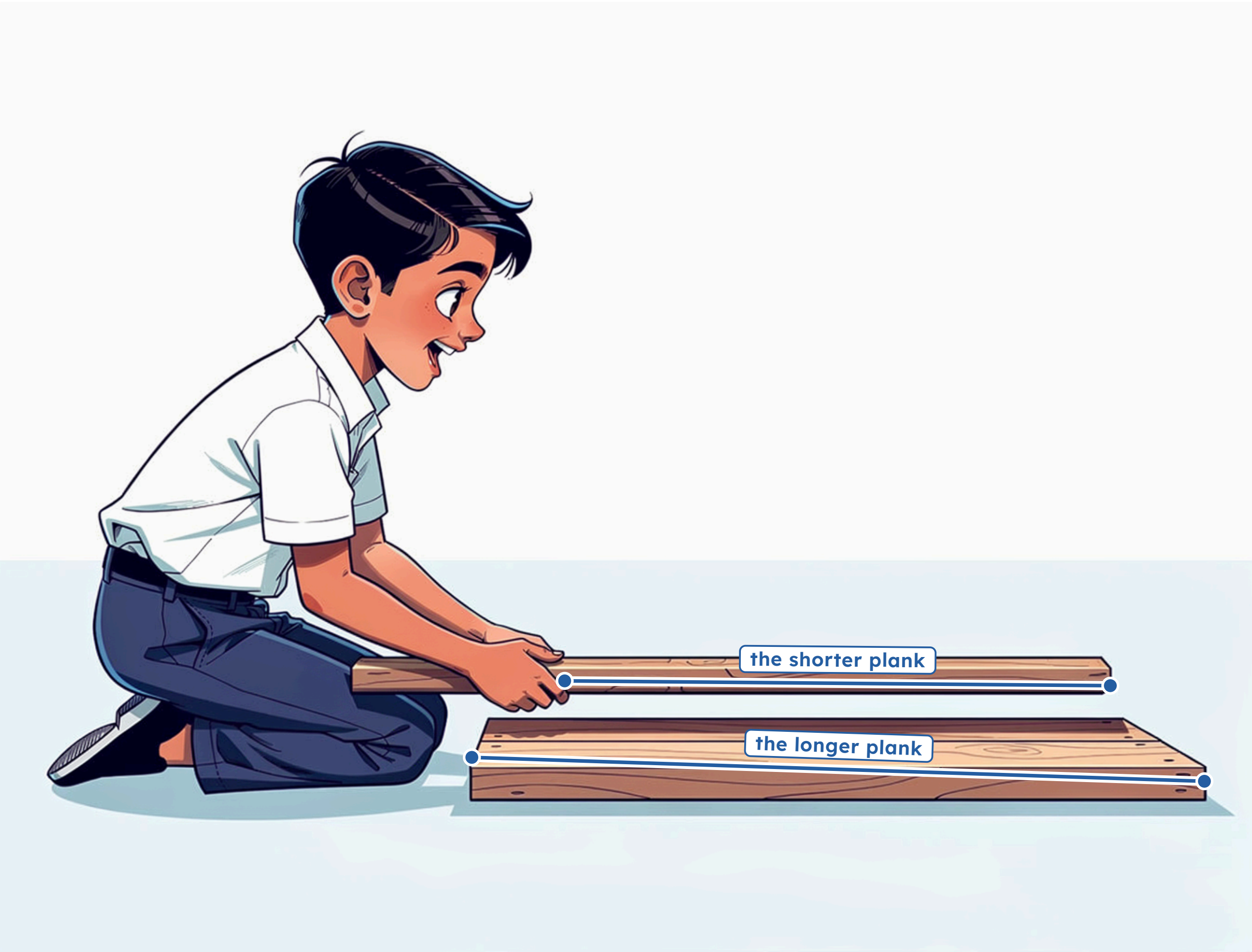
TRY
Find the HCF and LCM of 8 and 12.

Answer: HCF $= 4$ and LCM $= 24$.

CAUTION
Swap the two and both answers go wrong. Smaller powers make the HCF. Larger powers make the LCM.

FIND THE HCF AND LCM OF ANY TWO NUMBERS, FROM THEIR PRIMES.

1. **Factor both numbers** Write 12 and 18 as primes. $12 = 2^2 \cdot 3$. $18 = 2 \cdot 3^2$.
2. **List every prime** The primes that appear are 2 and 3.
3. **Take the smaller power** For 2, the powers are 2^2 and 2^1. Take the smaller, 2^1. For 3, the powers are 3^1 and 3^2. Take the smaller, 3^1. HCF $= 2^1 \cdot 3^1 = 6$.
4. **Take the bigger power** For 2, take the bigger power, 2^2. For 3, take the bigger power, 3^2. LCM $= 2^2 \cdot 3^2 = 36$.
5. **Check by multiplying** HCF $\cdot$ LCM $= 6 \cdot 36 = 216$. $12 \cdot 18 = 216$ too. They match, so the answer checks out.



Dhruv checks two wooden planks laid side by side, one clearly longer than the other.

CHECK YOURSELF

19. Factoring $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$, the HCF is found by

- **A.** taking 2^2 and 3^2, the larger powers, giving 36
- **B.** taking 2^1 and 3^2, mixing the two numbers' larger powers, giving 18
- **C.** taking 2^1 and 3^1, the smaller powers, giving 6
- **D.** adding the exponents of 2 and 3 across both numbers, giving $2^3 \cdot 3^3$

20. After computing $\mathrm{HCF}(12, 18) = 6$ and $\mathrm{LCM}(12, 18) = 36$, a check confirms $6 \cdot 36 = 12 \cdot 18$. If the check had instead given $6 \cdot 36 \neq 12 \cdot 18$, what would that mean?

- **A.** the product relation is only approximately true, so small mismatches are normal
- **B.** an arithmetic error was made in the factorisation or the HCF/LCM
- **C.** 12 and 18 would be an exception to the product relation
- **D.** the HCF and LCM would need to be added instead of multiplied to fix it

1 more in this set online.

WORKED EXAMPLE · Why powers of 9 never end in zero

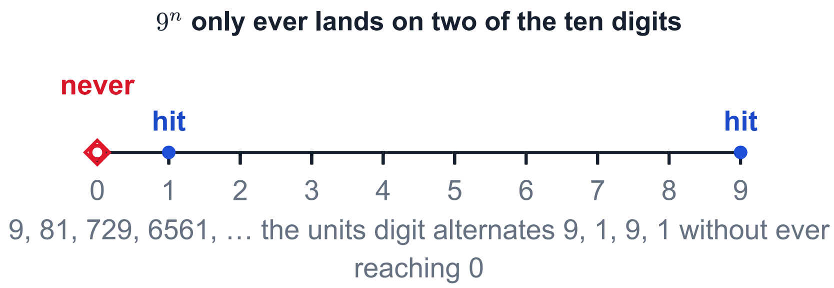
1.	assume 9^n ends in the digit 0	*Start from the ending we want to rule out. See what it would force to be true.*
2.	a number ending in 0 is divisible by $10 = 2 \cdot 5$	*Any number ending in 0 is a multiple of ten. Ten factors as 2 times 5.*
3.	so 9^n's factorisation would need both 2 and 5 in it	*If 10 divides 9^n, then 9^n's own factorisation must contain both of 10's prime factors.*
4.	$9 = 3^2$, so $9^n = 3^{2n}$	*9 is built from the single prime 3. So every power of 9 is too. No other prime can enter.*
5.	contradiction: the only prime in 9^n is 3	*The Fundamental Theorem of Arithmetic fixes which primes make up a number. 9^n can never also contain 2 and 5.*
6.	9^n never ends in 0, for any n	*The assumed ending forced something impossible. So the assumption fails.*
7.	$9^1 = 9$, $9^2 = 81$, $9^3 = 729$, $9^4 = 6561$	CHECK ***Check the pattern yourself.*** *None of these powers end in 0. That matches what you just proved for every n.*

BACK
Every composite number factors into primes, uniquely · p. 9

TRY
Can 9^n ever end in the digit 5?

Answer: No. Ending in 5 needs a factor of 5, and 9^n holds only 3s.

RULE
Ask which primes that last digit needs. Then check whether they can appear. Ending in 0 needs both a 2 and a 5.



The units digit alternates between 9 and 1 for every power of 9 shown, and 0 never appears.

DECIDE WHETHER A NUMBER CAN END IN A CHOSEN DIGIT, WITHOUT CHECKING EVERY POWER.

1. **Name the digit** We want to know if 9^n can end in 0.
2. **Find the needed primes** Ending in 0 needs a factor of $10 = 2 \cdot 5$. So the number needs both 2 and 5 in its primes.
3. **Factor the number** $9 = 3^2$, so $9^n = 3^{2n}$. Its only prime factor is 3.
4. **Compare and conclude** 3 is not 2 and not 5. So 9^n can never end in 0, for any n.

CHECK YOURSELF

21. **$9^n = 3^{2n}$ for every natural number n. This shows 9^n can never end in 0 because**

- **A.** ending in 0 needs both 2 and 5; 9^n only has 3
- **B.** 9^n is always odd, and no odd number can end in 0
- **C.** 9 itself does not end in 0, so no power of it can either
- **D.** n would need to be negative for 9^n to end in 0

22. **A student argues: “9^n ends in $9, 1, 9, 1, \ldots$ alternately, so it can never end in 0 — that is just a pattern, nothing to do with primes.” What does this argument miss?**

- **A.** the real reason is structural
- **B.** the pattern claim is false; 9^n does sometimes end in other digits
- **C.** the student is completely correct, and prime factorisation is not needed here
- **D.** the argument is right for even n but wrong for odd n

1 more in this set online.

WORKED EXAMPLE · A sum that factors is composite

1.	$3 \cdot 5 \cdot 7 + 7$	*Look for a factor every term already shares. Do not add the sum first.*
2.	$= 7 \cdot (3 \cdot 5 + 1)$	*7 divides every term on the left. So we can pull it out the front.*
3.	$= 7 \cdot 16 = 112$	*Finish the arithmetic inside the bracket. Then multiply.*
4.	112 is composite	*It is written as a product of two integers greater than 1. That is exactly what composite means. No further factor check is needed.*

WORKED EXAMPLE · A sum that factors is composite

5. $112 = 4 \cdot 28$

CHECK ***Check it a different way.*** *112 splits into another pair of factors greater than 1 too. That confirms it is composite by more than one route.*

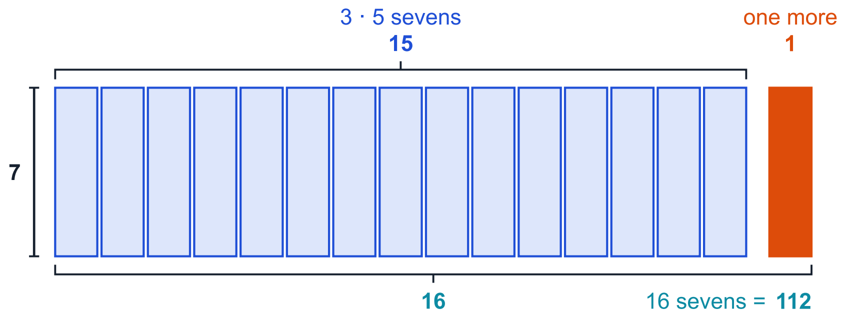
BACK
Prime and composite numbers · p. 6

TRY
Explain why $2 \cdot 3 \cdot 5 + 5$ is composite.

Answer: Every term shares 5: $2 \cdot 3 \cdot 5 + 5 = 5 \cdot (2 \cdot 3 + 1) = 5 \cdot 7 = 35$, composite.

CAUTION
Primes added together need not give a prime. Look for a shared factor first.

Fifteen sevens and one more make sixteen sevens



Every term of 3 · 5 · 7 + 7 is a stack of sevens, so the whole sum is 16 stacks of 7. A number that is 7 times 16 has a factor besides 1 and itself: it is composite before anyone checks a prime list.

A sum whose terms share a factor is that many copies of it, so $3 \cdot 5 \cdot 7 + 7$ is sixteen sevens, not prime.

SHOW A SUM IS COMPOSITE, WITHOUT CHECKING ANY FACTORS BY HAND.

1. **Find a shared factor** Every term in $3 \cdot 5 \cdot 7 + 7$ shares the factor 7.
2. **Pull it out front** $3 \cdot 5 \cdot 7 + 7 = 7 \cdot (3 \cdot 5 + 1)$.
3. **Finish the arithmetic** $3 \cdot 5 + 1 = 16$. So the sum is $7 \cdot 16 = 112$.
4. **Name it composite** 112 is a product of two integers greater than 1. So 112 is composite.

CHECK YOURSELF

23. $3 \cdot 5 \cdot 7 + 7$ **is shown to be composite by**

- **A.** computing the sum directly and checking it against a list of primes
- **B.** noticing that 7 appears in the expression, which alone makes it composite
- **C.** factoring out the shared 7: $7 \cdot (3 \cdot 5 + 1) = 7 \cdot 16$
- **D.** dividing the whole expression by 3 instead of 7

24. A student says: “this number is composite just because it is a sum, and sums are never prime.” What is wrong?

- **A.** the student is right — every sum of integers is composite
- **B.** the number is actually prime, and the worked example is wrong
- **C.** sums CAN be prime (for example $2 + 3 = 5$)
- **D.** the factor that matters is 3, not 7

1 more in this set online.

WORKED EXAMPLE · Root 3 is irrational, the same way

1.	assume, to the contrary, $\sqrt{3} = a/b$ with a, b coprime	*This is proof by contradiction. It reuses the $\sqrt{2}$ proof's method with the prime 3.*
2.	$a^2 = 3b^2$	*Square both sides. Then clear the denominator.*
3.	3 divides a^2, so 3 divides a	*The same lemma as before. Apply it with $p = 3$.*
4.	$a = 3c$, so $9c^2 = 3b^2$, so $b^2 = 3c^2$	*Substitute $a = 3c$. $(3c)^2 = 9c^2$, so $9c^2 = 3b^2$. Divide both sides by 3.*
5.	3 divides b^2, so 3 divides b	*The same lemma again.*
6.	3 divides both a and b	*But a and b were assumed coprime. This is the same contradiction as the $\sqrt{2}$ proof, one prime later.*
7.	$\sqrt{3}$ is irrational	*The assumption fails. So its opposite holds.*
8.	$\sqrt{3} \approx 1.7320508$	CHECK ***Check it for yourself.*** *This decimal never settles into a repeating block. That is exactly what an irrational number's expansion looks like.*

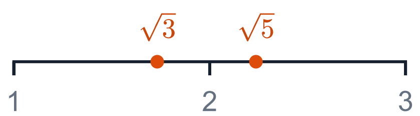
BACK
Root 2 cannot be written as a fraction · p. 14

TRY
Which prime plays the role that 2 played in the $\sqrt{2}$ proof?

Answer: 3. It replaces 2 at every step.

RULE
Put 3 wherever the $\sqrt{2}$ proof used 2. Every step still works.

Two more points that miss every grid line



neither point has an integer or a simple fraction sitting on it

The same picture holds for the square root of any prime, not only 3 and 5.

CHECK YOURSELF

25. The $\sqrt{3}$ proof reuses the $\sqrt{2}$ proof's method by

- **A.** substituting the prime 3 everywhere 2 appeared
- **B.** starting completely from scratch with a different technique
- **C.** using 3 as a coprime pair with 2
- **D.** checking a few fractions close to $\sqrt{3}$

26. Could the same method prove $\sqrt{4}$ irrational, by substituting 4 for the prime in the argument?

- **A.** yes — the method works for substituting any integer, prime or not
- **B.** yes, but only because 4 is a perfect square
- **C.** no, because $\sqrt{4} = 2$ is itself prime
- **D.** no — 4 is not prime

1 more in this set online.

WORKED EXAMPLE · Root 5 is irrational, the same way

1.	assume, to the contrary, $\sqrt{5} = a/b$ with a, b coprime	*This is proof by contradiction. It reuses the same method with the prime 5.*
2.	$a^2 = 5b^2$	*Square both sides. Then clear the denominator.*
3.	5 divides a^2, so 5 divides a	*The same lemma again. Apply it with $p = 5$.*
4.	$a = 5c$, so $25c^2 = 5b^2$, so $b^2 = 5c^2$	*Substitute $a = 5c$. $(5c)^2 = 25c^2$, so $25c^2 = 5b^2$. Divide both sides by 5.*
5.	5 divides b^2, so 5 divides b	*The same lemma again.*
6.	5 divides both a and b	*But a and b were assumed coprime. This is the same contradiction, this time with the prime 5.*
7.	$\sqrt{5}$ is irrational	*The assumption fails. So its opposite holds.*
8.	$\sqrt{5} \approx 2.2360679$	CHECK ***Check it for yourself.*** *This decimal never settles into a repeating block either. That is the same signature you just saw for $\sqrt{3}$.*

CAUTION
Losing that square is the commonest slip here. Check it before you move on.

RULE
$a = 5c$ squares to $25c^2$, not $5c^2$. Square the 5 as well.

CHECK YOURSELF

27. In the $\sqrt{5}$ proof, squaring a/b gives $a^2 = 5b^2$. The next step concludes 5 divides a because

- **A.** a^2 is always a multiple of 5 when b is an integer
- **B.** $5b^2$ is even, so a must be even too
- **C.** a was already assumed to be a multiple of 5 from the start
- **D.** 5 is prime and 5 divides a^2

28. A student writes $a = 5c$ and then claims $25c^2 = 5b^2$ directly gives $c^2 = 5b^2$. What is the error?

- **A.** there is no error; both forms are equivalent
- **B.** the error is in writing $a = 5c$ in the first place
- **C.** dividing by 5 gives $b^2 = 5c^2$, not $c^2 = 5b^2$
- **D.** the equation should have been divided by 25, not 5

WORKED EXAMPLE · 3 plus 2 root 5 is irrational

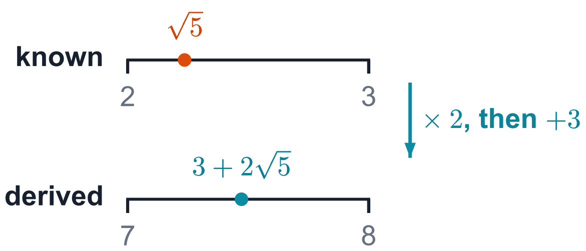
1.	assume, to the contrary, $3 + 2\sqrt{5} = r$ for some rational r	*This is proof by contradiction. Assume the opposite of what we want to prove.*
2.	$2\sqrt{5} = r - 3$	*Rearrange. A difference of two rationals is rational.*
3.	$\sqrt{5} = (r - 3)/2$	*Divide by 2. A rational number divided by a nonzero integer is still rational.*
4.	but $\sqrt{5}$ is irrational	*We already proved this.*
5.	contradiction, so $3 + 2\sqrt{5}$ is irrational	*The assumption forces $\sqrt{5}$ to be rational. That is false. So the assumption itself must be false.*
6.	$3 + 2 \cdot 2.2360679 \approx 7.4721359$	CHECK ***Check it yourself.*** *This decimal never settles into a repeating block either. That is exactly what the algebra above just proved.*

TRY
Why would $2\sqrt{5}$ be rational if $3 + 2\sqrt{5}$ were rational?

Answer: A rational number minus the rational 3 is still rational.

RULE
Get $\sqrt{5}$ on its own first. Then use the proof you have already done.

One position, carried through $\times 2, +3$



The same picture works for any number built by multiplying a known irrational by a rational number and adding another.

CHECK YOURSELF

29. To prove $3 + 2\sqrt{5}$ is irrational, the proof assumes it equals a rational r and then

- **A.** repeats the full coprime-contradiction argument from scratch on $3 + 2\sqrt{5}$
- **B.** isolates $\sqrt{5}$ algebraically, showing it would have to be rational too
- **C.** checks a few decimal values of $3 + 2\sqrt{5}$
- **D.** assumes 3 and $2\sqrt{5}$ are each irrational separately

30. The step "$2\sqrt{5} = r - 3$, so it is rational" relies on which fact?

- **A.** a difference of two rational numbers is always rational
- **B.** a difference of a rational and an irrational number is always rational
- **C.** any number that can be isolated algebraically is automatically rational
- **D.** $r - 3$ is rational only when r is a whole number

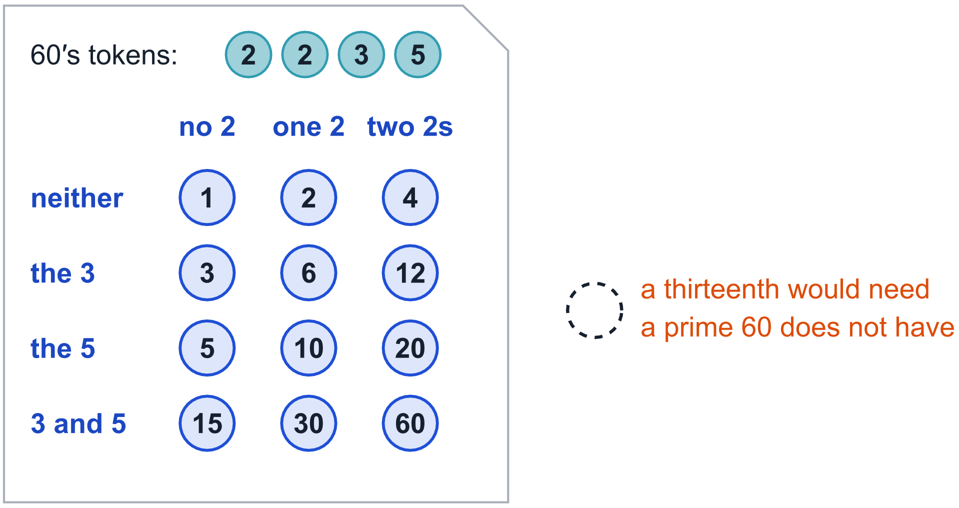
1 more in this set online.

Wrapping up

IN EVERY CHAPTER

Recap

Twelve tray sizes, and no thirteenth



Each tray size is a choice from 60's four tokens: how many 2s, whether the 3, whether the 5. Three times two times two is twelve, and a thirteenth would need a token that is not there.

A grid of twelve discs comes from three choices for the 2s, two for the 3, and two for the 5.

- FTA: every number breaks into primes in one way. $1400 = 2^3 \cdot 5^2 \cdot 7$. No other set of primes works.
- HCF AND LCM: read them off two factorisations. For 8 and 12: HCF $= 4$ and LCM $= 24$.
- PRODUCT RELATION: $\mathrm{HCF}(a, b) \cdot \mathrm{LCM}(a, b) = a \cdot b$. Check it: $4 \cdot 24 = 96$, and $8 \cdot 12 = 96$ too.
- PRIME-SQUARE LEMMA: if a prime p divides a^2, then p divides a. For $a = 6$, $p = 2$ divides both 36 and 6.
- CONTRADICTION TECHNIQUE: assume $\sqrt{2} = a/b$ in lowest terms. The argument forces 2 to divide both a and b. That is impossible, since coprime numbers share no common factor.

You have used every one of these five rules already, in the pages above. We built each one from ONE tool: a number's prime factorisation never changes. So an assumption that forces a fraction's numerator and denominator to keep sharing a factor forever must be false. This is what proves $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ irrational. Simple algebra then extends it to numbers such as

$3+2\sqrt{5}$. Look back at the five rules above. Notice how often the same numbers turn up: 8, 12 and 6 appear in more than one rule.

And the question this chapter opened with. How many tray sizes split 60 exactly? Every tray size has to be built out of 60's own primes, 2, 2, 3 and 5, and nothing else — that is the Fundamental Theorem again. Take any combination of them and you get 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60. Twelve sizes. There is no thirteenth, because a thirteenth would need a prime that 60 does not have.

KEY
Two ideas do all the work. Every number has one prime factorisation. Assume the opposite, then reach something impossible.

BACK
Every composite number factors into primes, uniquely · p. 9

RULE
To show a new number is irrational, turn it into one you have already done. $3+2\sqrt{5}$ turns into $\sqrt{5}$.

CHECK YOURSELF

31. Every proof in this chapter — for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $3+2\sqrt{5}$ — ultimately traces back to

- **A.** a separate rule for each individual number proved irrational
- **B.** the specific decimal expansions of each square root
- **C.** the Fundamental Theorem of Arithmetic's uniqueness
- **D.** a general rule that all square roots are irrational

32. Why does an assumption that "a fraction's numerator and denominator keep sharing a factor forever" have to be false?

- **A.** because fractions are not allowed to share any factors, by definition
- **B.** because the assumption is only false for even numbers
- **C.** because a number's prime factorisation is fixed and finite
- **D.** because every fraction is already in lowest terms automatically

1 more in this set online.

CHECK YOURSELF: THE WHOLE CHAPTER

33. Every tool in this chapter — HCF, LCM, digit endings, and square-root proofs — comes from one fact. What is that fact?

- **A.** Every prime number is itself irrational, so it can prove other numbers irrational too. Being irrational is treated as the useful property here.
- **B.** Every integer greater than 1 factors into primes in exactly one way, apart from the order of the factors.
- **C.** There are infinitely many prime numbers, so a proof can always find a new one to use.
- **D.** Every composite number is bigger than every prime number that divides it.

34. According to the Fundamental Theorem of Arithmetic, $1400 = 2^3 \cdot 5^2 \cdot 7$. What does this theorem guarantee about 1400?

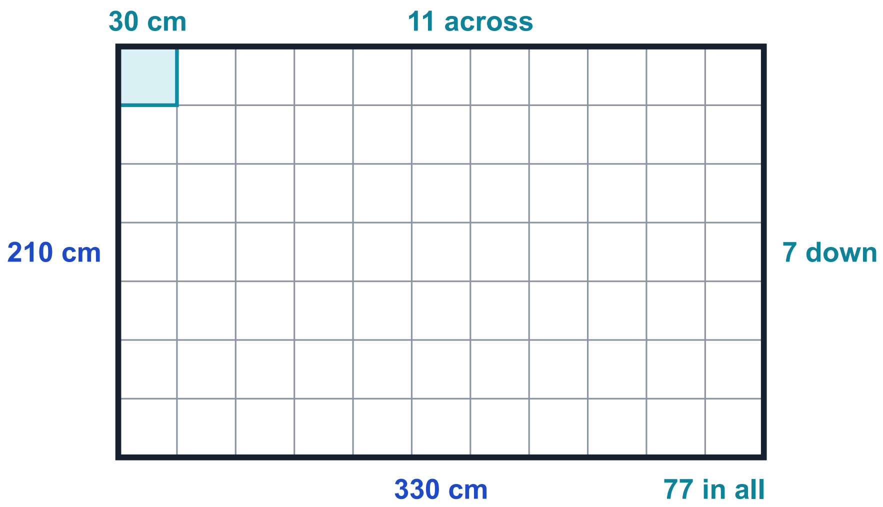
- **A.** 1400 can also be written as a product of primes in a completely different set, such as using 11 or 13 instead.
- **B.** 1400 has more than one correct prime factorisation, and either one may be used.
- **C.** No other set of primes, in any combination, multiplies to give 1400 — this is the only prime factorisation it has.
- **D.** 1400 itself must be a prime number, since its factorisation uses only primes.

3 more in this set online.

IN EVERY CHAPTER

Where you will meet this

The biggest tile that needs no cutting



Any bigger square tile would have to be cut at an edge.

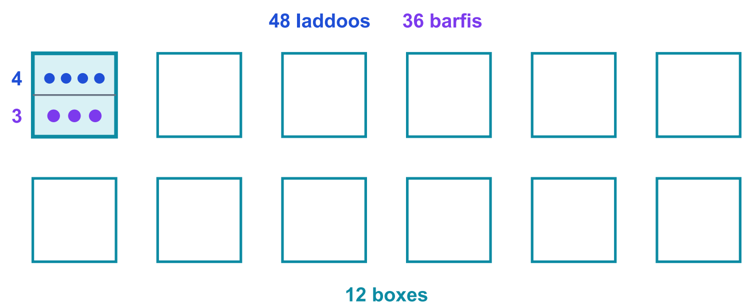
The same reading works for any two lengths a problem gives you: factor both, and the HCF is the biggest tile that leaves nothing over.

You may not write another proof after Class 10. But primes, HCF, LCM and square roots stay with you. You will meet them in shops, on floors and on your phone. Here are eight of those places.

- **Tiling a verandah.** Your verandah is 330 cm by 210 cm. You want the biggest square tile that needs no cutting.
  The maths: the side of that tile is the HCF of the two lengths.
  $330 = 2 \cdot 3 \cdot 5 \cdot 11$ and $210 = 2 \cdot 3 \cdot 5 \cdot 7$. So the HCF is 30 cm, and you need $11 \times 7 = 77$ tiles.

BACK
A rational number in lowest terms · p. 7

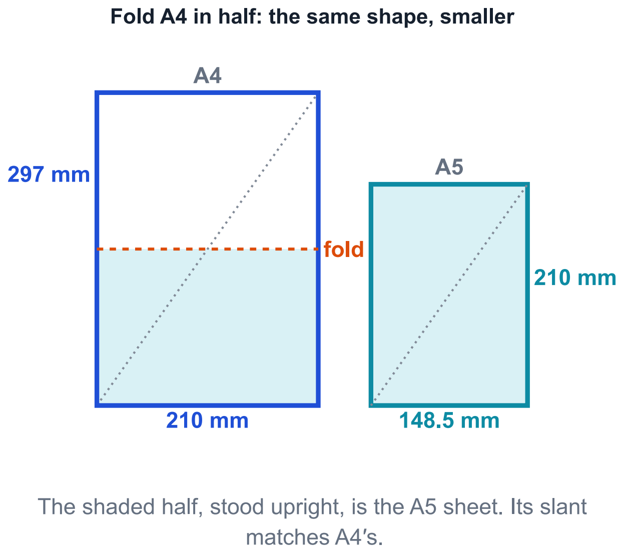
The most boxes with nothing left over



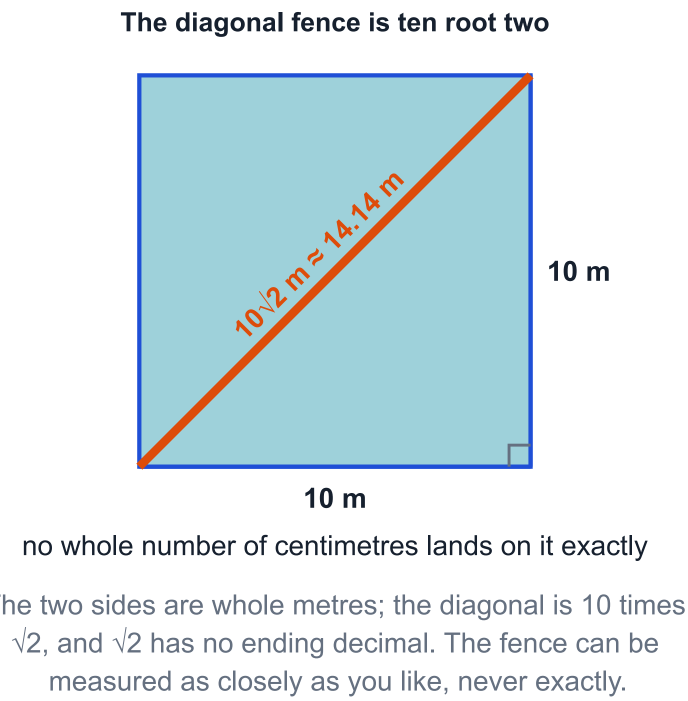
Every box is packed the same. Only the first is shown open.

The HCF answers the other question too: not how big each equal part is, but how many equal parts two amounts will make.

- **Packing sweet boxes for Diwali.** Your shop has 48 laddoos and 36 barfis. Every box must be the same, with nothing left over.
  The maths: the most boxes you can make is the HCF of 48 and 36.
  $48 = 2^4 \cdot 3$ and $36 = 2^2 \cdot 3^2$. So the HCF is $2^2 \cdot 3 = 12$ boxes, each with 4 laddoos and 3 barfis.
- **Buying for a party.** Paper plates come in packs of 8. Cups come in packs of 12. You want the same number of each, with none spare.
  The maths: the smallest such number is the LCM of 8 and 12.
  $8 = 2^3$ and $12 = 2^2 \cdot 3$. So the LCM is $2^3 \cdot 3 = 24$: three packs of plates and two packs of cups.



Halving keeps the shape only when the long side is root two times the short one, and no whole number of millimetres hits that exactly.



A square plot with 10 m sides has a diagonal fence of 10 root 2 m, about 14.14 m.

- **Paying two plans.** Your phone plan renews every 28 days. Your streaming plan renews every 30 days. Both renewed today.
  The maths: both fall due on the same day again after the LCM of 28 and 30 days.
  $28 = 2^2 \cdot 7$ and $30 = 2 \cdot 3 \cdot 5$. So the LCM is $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$ days, a little over a year.

- **Jogging with a friend.** You run one lap of the park in 6 minutes. Your friend takes 8 minutes. You start together.
  The maths: you cross the start line together again after the LCM of 6 and 8 minutes.
  $6 = 2 \cdot 3$ and $8 = 2^3$. So the LCM is $2^3 \cdot 3 = 24$ minutes. That is your fourth lap and your friend's third.
- **Paying online.** Many banking and shopping sites lock your card details with one huge number. That number is two giant primes multiplied together.
  The maths: every whole number above 1 has exactly one prime factorisation. For a huge number, finding it is very slow.
  Multiplying is quick: $7 \cdot 11 = 77$. Undoing it for a number over 600 digits long would take far longer than a lifetime.
- **Choosing a paper size.** Fold an A4 sheet in half and you get A5. It has the same shape as A4, only smaller.
  The maths: that works only when the long side is $\sqrt{2}$ times the short side. And $\sqrt{2}$ is an irrational number.
  A4 is 297 mm by 210 mm, and $297/210 \approx 1.414$. No sheet in whole millimetres can hit $\sqrt{2}$ exactly.
- **Fencing a square plot.** Your plot is a square, 10 m on each side. You want a fence along its diagonal.
  The maths: the diagonal is $10\sqrt{2}$ m. Since $\sqrt{2}$ is irrational, its decimal never ends.
  $10^2 + 10^2 = 200$, so the diagonal is $\sqrt{200} = 10\sqrt{2} \approx 14.14$ m.

Your turn. You are setting out chairs for a wedding. There are 42 chairs for the bride's guests and 30 for the groom's. Every row must be the same length, and no row mixes the two sides. What is the longest row you can make? *Answer:* $42 = 2 \cdot 3 \cdot 7$ and $30 = 2 \cdot 3 \cdot 5$, so the HCF is 6. That gives 7 rows for the bride's side and 5 for the groom's.

Practice: Exercise 1.1

PRACTICE SET

1. PRACTICE Find the prime factorisation of 594. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1.	$594/2 = 297$	*divide by the SMALLEST prime that goes in; 594 is even, so that prime is 2*
2.	297 is odd, so no second 2 comes out; $297/3 = 99$, then $99/3 = 33$, then $33/3 = 11$	*297's digits add to 18, a multiple of 3, so 3 goes in — and it keeps going in, three times over*
3.	$11/11 = 1$, and the chain stops	*11 is prime, so only itself divides it; reaching 1 is the signal that nothing is left to break up*
4.	collect the divisors used, in the order they came out — $594 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 11 = 2 \cdot 3^3 \cdot 11$	*each division took out exactly one prime, so the divisors used ARE the factorisation*

SOLUTION — Q1 · Worked in full

5.	the check — multiply back, $3^3 = 27$ and $2 \cdot 27 \cdot 11 = 594$	*if the product does not come back to the number you started with, one division went wrong; this costs one line and catches it*

2. PRACTICE Find the HCF and LCM of 66 and 110 by prime factorisation, and check that $\text{HCF} \cdot \text{LCM} = 66 \cdot 110$. (Factor each one the same way as above. Then take the smaller power of every prime they share for the HCF, and the larger power of every prime you see for the LCM.)

3. PRACTICE Find the HCF and LCM of 20, 28 and 35 by prime factorisation. (Three numbers this time. The prime method does not change at all — but the $\text{HCF} \cdot \text{LCM}$ rule from the last item is for two numbers only, so it is not available here.)

4. PRACTICE Find the prime factorisation of 720.

5. PRACTICE Find the HCF and LCM of 96 and 120 by prime factorisation.

6. PRACTICE Find the HCF and LCM of 12, 18 and 24 by prime factorisation.

7. PRACTICE Two buses leave a bus stand together. One returns to the stand every 24 minutes, the other every 36 minutes. After how many minutes will they next leave together?

8. PRACTICE Using the Fundamental Theorem of Arithmetic, show that 7^n can never end in the digit 5, for any natural number n.

9. PRACTICE Show that $6 \cdot 7 \cdot 11 + 11$ is a composite number.

ANSWERS **1.** $594 = 2 \cdot 3^3 \cdot 11$.
2. $66 = 2 \cdot 3 \cdot 11$ and $110 = 2 \cdot 5 \cdot 11$, so $\text{HCF} = 2 \cdot 11 = 22$ and $\text{LCM} = 2 \cdot 3 \cdot 5 \cdot 11 = 330$; the check gives $22 \cdot 330 = 7260 = 66 \cdot 110$.
3. $20 = 2^2 \cdot 5$, $28 = 2^2 \cdot 7$, $35 = 5 \cdot 7$; no prime is in all three, so $\text{HCF} = 1$, and $\text{LCM} = 2^2 \cdot 5 \cdot 7 = 140$.
4. $720 = 2^4 \cdot 3^2 \cdot 5$. **5.** $\text{HCF} = 24$, $\text{LCM} = 480$. **6.** $\text{HCF} = 6$, $\text{LCM} = 72$.
7. 72 minutes.
8. 7^n's only prime factor is 7, so 5 can never divide it — it can never end in 5.
9. $6 \cdot 7 \cdot 11 + 11 = 11 \cdot 43 = 473$, a product of two integers greater than 1.
Full solutions: **manthika.com/learn/math10-ch01**

Further practice

PRACTICE SET

10. PRACTICE Find the prime factorisation of 3675.

SOLUTION — Q10 · Worked in full

1.	$3675 = 3 \cdot 1225$	*divide out the smallest prime that goes in, 3*
2.	$1225 = 5 \cdot 245$	*divide by 5*
3.	$245 = 5 \cdot 49$	*divide by 5 again*
4.	$49 = 7^2$	*49 is 7 times 7*

SOLUTION — Q10 · **Worked in full**

5. $3675 = 3 \cdot 5^2 \cdot 7^2$ — *collect the primes found at each division*

11. PRACTICE Find the prime factorisation of 1050.
12. PRACTICE Find the HCF and LCM of 45 and 75 by prime factorisation, and verify that $\mathrm{HCF}(45, 75) \cdot \mathrm{LCM}(45, 75) = 45 \cdot 75$.
13. PRACTICE Find the HCF and LCM of 84 and 126 by prime factorisation.
14. PRACTICE Find the HCF and LCM of 15, 20 and 45, using prime factorisation.
15. PRACTICE Given that $\mathrm{HCF}(180, 252) = 36$, find $\mathrm{LCM}(180, 252)$.
16. PRACTICE Three bells ring at intervals of 20, 24 and 30 minutes. If all three ring together at eight in the morning, after how many minutes will they next ring together?
17. PRACTICE Show that 15^n can never end in the digit 0, for any natural number n.
18. PRACTICE Show that $5 \cdot 7 \cdot 11 \cdot 13 + 11 \cdot 13$ is a composite number.
19. PRACTICE The HCF of two positive integers is 4 and their product is 240. What is their LCM?
    A. 40
    B. 48
    C. 60
    D. 96

ANSWERS **10.** $3675 = 3 \cdot 5^2 \cdot 7^2$. **11.** $1050 = 2 \cdot 3 \cdot 5^2 \cdot 7$. **12.** HCF $= 15$, LCM $= 225$. **13.** HCF $= 42$, LCM $= 252$. **14.** HCF $= 5$, LCM $= 180$. **15.** $\mathrm{LCM}(180, 252) = 1260$.
16. 120 minutes.
17. 15^n's only prime factors are 3 and 5, so 2 never divides it — it can never end in the digit 0.
18. $5 \cdot 7 \cdot 11 \cdot 13 + 11 \cdot 13 = 11 \cdot 13 \cdot (5 \cdot 7 + 1) = 143 \cdot 36 = 5148$, a product of two integers greater than 1.
19. C — 60.
Full solutions: **manthika.com/learn/math10-ch01**

Practice: Exercise 1.2

PRACTICE SET

1. PRACTICE Prove that $\sqrt{11}$ is irrational. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1. assume, to the contrary, that $\sqrt{11} = a/b$ for integers a and $b \neq 0$, with a and b coprime — *every rational number can be written in lowest terms, so this is what $\sqrt{11}$ would look like if it were rational*
2. square both sides and clear the fraction — $11 = a^2/b^2$, so $a^2 = 11b^2$ — *squaring is what turns the root into a whole number the lemma can be used on*

SOLUTION — Q1 · Worked in full

3.	11 divides a^2, so 11 divides a; write $a = 11c$	*11 is prime, so the lemma applies — a prime dividing a square divides the number itself*
4.	$(11c)^2 = 11b^2$ gives $121c^2 = 11b^2$, so $b^2 = 11c^2$, so 11 divides b	*the same lemma a second time; do not stop after the first use, because one shared factor is not yet a contradiction*
5.	11 divides both a and b, which contradicts a and b being coprime, so $\sqrt{11}$ is irrational	*the assumption forced something impossible, so the assumption was false*
6.	the check — the step that needs 11 to be PRIME is the third one; run it with 4 instead and it breaks, because 4 divides $6^2 = 36$ but 4 does not divide 6	*and it SHOULD break, since $\sqrt{4} = 2$ is rational; that one line tells you the proof works for primes only, and why*

2. PRACTICE Prove that $5 + \sqrt{11}$ is irrational. (Get $\sqrt{11}$ on its own first. Then use what item 1 has just proved, rather than starting again.)

3. PRACTICE Prove that $3/\sqrt{11}$ is irrational. (One step further — 3 divided by a nonzero rational is still rational.)

4. PRACTICE Prove that $\sqrt{5}$ is irrational.

5. PRACTICE Prove that $3 + 2\sqrt{5}$ is irrational.

6. PRACTICE Prove that $1/\sqrt{2}$ is irrational.

7. PRACTICE Prove that $7\sqrt{5}$ is irrational.

8. PRACTICE Prove that $6 + \sqrt{2}$ is irrational.

ANSWERS

1. Assume $\sqrt{11} = a/b$ with a, b coprime. Squaring gives $a^2 = 11b^2$, so 11 divides a^2 and hence a. Writing $a = 11c$ leads to $b^2 = 11c^2$, so 11 divides b too — contradicting that a and b are coprime.
2. Assume $5 + \sqrt{11} = r$ is rational; then $\sqrt{11} = r - 5$ would be a difference of two rationals, so rational — but $\sqrt{11}$ is irrational, a contradiction.
3. Assume $3/\sqrt{11} = r$ is rational, and $r \neq 0$; then $\sqrt{11} = 3/r$ would be rational, since 3 divided by a nonzero rational is rational — but $\sqrt{11}$ is irrational, a contradiction.
4. Assume $\sqrt{5} = a/b$ with a, b coprime; the same contradiction as the worked example — 5 divides both a and b.
5. Assume $3 + 2\sqrt{5}$ is rational; then $\sqrt{5}$ would be rational too, which is false.
6. Assume $1/\sqrt{2} = r$ is rational; then $\sqrt{2} = 1/r$ would be rational, since the reciprocal of a nonzero rational is rational — but $\sqrt{2}$ is irrational.
7. Assume $7\sqrt{5} = r$ is rational; then $\sqrt{5} = r/7$ would be rational — but $\sqrt{5}$ is irrational.
8. Assume $6 + \sqrt{2} = r$ is rational; then $\sqrt{2} = r - 6$ would be rational, as a difference of two rationals — but $\sqrt{2}$ is irrational.

Full solutions: **manthika.com/learn/math10-ch01**

BACK
Root 2 cannot be written as a fraction · p. 14

Further practice

PRACTICE SET

9. PRACTICE Prove that $2+\sqrt{3}$ is irrational.

10. PRACTICE Prove that $5\sqrt{2}$ is irrational.

11. PRACTICE Prove that $1/\sqrt{3}$ is irrational.

12. PRACTICE Prove that $4-\sqrt{5}$ is irrational.

13. PRACTICE Prove that $2/\sqrt{5}$ is irrational.

14. PRACTICE Prove that $\sqrt{7}$ is irrational.

15. PRACTICE Given that $\sqrt{7}$ is irrational, prove that $3\sqrt{7}$ is irrational.

ANSWERS

9. Assume $2+\sqrt{3}$ is rational; then $\sqrt{3}=\left(2+\sqrt{3}\right)-2$ would be a difference of two rationals, so rational — but $\sqrt{3}$ is irrational, a contradiction.
10. Assume $5\sqrt{2}=r$ is rational; then $\sqrt{2}=r/5$ would be rational, since a nonzero rational divided by 5 is rational — but $\sqrt{2}$ is irrational, a contradiction.
11. Assume $1/\sqrt{3}=r$ is rational; then $\sqrt{3}=1/r$ would be rational, since the reciprocal of a nonzero rational is rational — but $\sqrt{3}$ is irrational, a contradiction.
12. Assume $4-\sqrt{5}=r$ is rational; then $\sqrt{5}=4-r$ would be a difference of two rationals, so rational — but $\sqrt{5}$ is irrational, a contradiction.
13. Assume $2/\sqrt{5}=r$ is rational; then $\sqrt{5}=2/r$ would be rational, since 2 divided by a nonzero rational is rational — but $\sqrt{5}$ is irrational, a contradiction.
14. Assume $\sqrt{7}=a/b$ with a, b coprime. Squaring gives $a^2=7b^2$, so 7 divides a^2 and hence a. Writing $a=7c$ leads to $b^2=7c^2$, so 7 divides b too — contradicting that a and b are coprime.
15. Assume $3\sqrt{7}=r$ is rational; then $\sqrt{7}=r/3$ would be rational, since a nonzero rational divided by 3 is rational — but $\sqrt{7}$ is irrational, a contradiction.
Full solutions: **manthika.com/learn/math10-ch01**