CHAPTER 3

3 Pair of Linear Equations in Two Variables

WHAT THIS CHAPTER BUILDS

You will learn to solve two linear equations together, by drawing two lines and by algebra, and to tell in advance whether they meet once, never, or everywhere.

ALGEBRA UNIT, WORD PROBLEMS

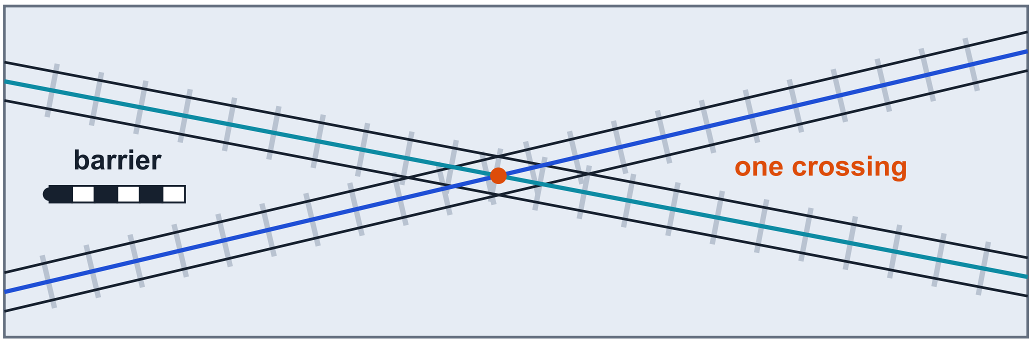
IN THIS CHAPTER

Tara and Arjun watch two railway tracks cross at one point and run on past each other.

Getting started

Where two lines meet

Two tracks, one crossing



side by side: never | on top of each other: every point

Two straight tracks that come in from different directions meet at one point and never again. Tracks side by side would never meet; tracks laid on top of each other would meet everywhere.

A pair of lines that are not parallel and not the same line share exactly one point, and that point is the pair's answer.

A STORY

Where the two tracks meet

Tara and Arjun stand by the railway, a barrier beside them. Two tracks come in from different directions. They cross once, then run on past each other.

“Only one crossing,” Arjun says, pointing at the tracks. “What if the tracks ran side by side instead? Would they ever cross?”

“Never,” Tara says. “Two lines like that stay the same distance apart forever.”

“And if the tracks were laid exactly on top of each other?” Arjun asks.

“Then every point is a crossing,” Tara says. “Infinitely many crossings, not one.”

“So two straight lines only ever give one of three answers,” Arjun says. “One crossing, no crossing, or every point.”

“Yes,” Tara says. “And you can tell which one, before you draw a single line.”

Which of the three would your own two equations give, before you plot either line?

Let us take the pair $x + y = 5$ and $x - y = 1$. Plot both as straight lines on the same graph. They meet at exactly one point.

Now nudge the second line, in your mind, until it runs exactly parallel to the first. It never meets the first line again. That gives no solution.

Slide it once more, so it lands exactly on top of the first line. Every point on that line is now a shared solution. That gives infinitely many solutions.

A pair of linear equations in x and y is nothing but two such lines. It always lands on exactly one of three outcomes: one solution, no solution, or infinitely many.

Try it yourself. Sketch $x + y = 5$ and $x - y = 1$ on paper, and find where they cross. (Answer: the point $(3, 2)$.)

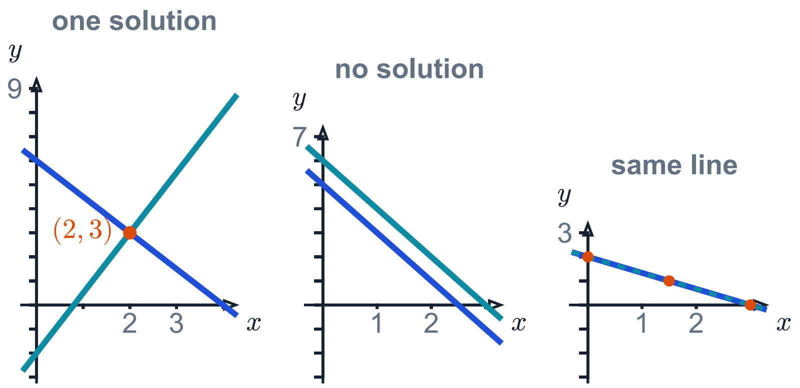
Which of the three holds can be decided before you plot a single point. We compare three ratios instead: a_1/a_2, b_1/b_2, and c_1/c_2, read straight off the two equations' own coefficients. This chapter builds that comparison first, then two algebraic methods that solve any such pair directly, and closes on situational problems that turn ordinary words into exactly this kind of pair.

Every method in this chapter finds the same point: wherever the two lines actually meet.

RULE
Compare the ratios a_1/a_2, b_1/b_2, c_1/c_2 before you solve. They tell you which case to expect.

KEY
The whole chapter asks one question: does a pair of lines meet once, never, or always, and if once, exactly where?

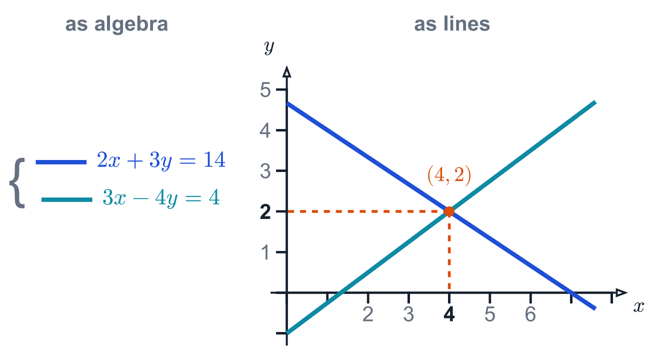
Cross once, stay parallel, or coincide



Third panel: the second equation is drawn dashed directly over the first, at the same coordinates — not a missing line, a different equation occupying the exact same one. The three marked points are shared solutions, sampled, not a complete list.

Three pairs of lines are shown side by side: one crosses once, one never meets, and one is the same line drawn twice.

A pair of equations is a pair of lines



Putting $x = 4$ and $y = 2$ into both: $2(4) + 3(2) = 14$, and $3(4) - 4(2) = 4$.

The crossing is read straight off the axes, and the line underneath puts those two numbers back into both equations as a check.

IN THE WILD

One cab charges a higher base fare and a lower rate per kilometre. The other charges the reverse. Draw each cab's cost against distance, and you get two straight lines. The ratios of their coefficients say, before either line is drawn, whether the lines cross once, never, or are the same line twice. That tells you, before any ride, the distance past which the second cab is cheaper, or that it never is.

CHECK YOURSELF

1. A pair of linear equations in x and y is really a pair of straight lines. How many outcomes are possible for such a pair?

- **A.** always exactly one solution, since two lines must cross somewhere
- **B.** either no solution or infinitely many, never a single point
- **C.** as many solutions as there are values of x
- **D.** exactly three: one solution, no solution, or infinitely many

2. Before drawing either line, what tells you whether a pair of linear equations will have one solution, none, or infinitely many?

- **A.** comparing the signs of a_1 and a_2 alone
- **B.** comparing the ratios a_1/a_2, b_1/b_2, c_1/c_2
- **C.** comparing which equation has the larger constant term c
- **D.** counting how many terms each equation has

1 more in this set online.

IN EVERY CHAPTER

Before you start

DO THIS FIRST

You have met every tool you need here, one at a time. Now two lines share one graph. Try each check below. It takes a minute.

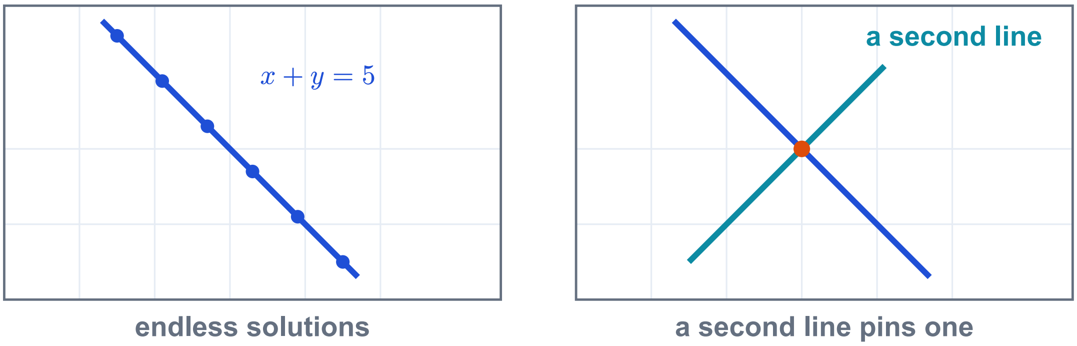
- **Coordinates of a point** (Class 9, Ganita Manjari, page 6).
  Here: every solution of an equation like $x + y = 5$ is a point you can plot.
  Check: $(2, 3)$ is on the line $x + y = 5$, since $2 + 3 = 5$.
- **A line from two points** (Class 9, Ganita Manjari, page 27).
  Here: the graphical method draws each equation from two solutions.
  Check: On $y = 2x + 1$, $x = 0$ gives $y = 1$ and $x = 3$ gives $y = 7$.
- **The slope of a line** (Class 9, Ganita Manjari, page 31).
  Here: two different lines with the same slope never cross.
  Check: $y = 2x + 1$ and $y = 2x - 1$ stay 2 apart at every x.
- **Solving a linear equation** (Class 9, Ganita Manjari, page 20).
  Here: substitution and elimination each end in one such equation.
  Check: $20 - 3y = 5$ gives $3y = 15$, so $y = 5$.
- **Words into an equation** (Class 9, Ganita Manjari, page 20).
  Here: a word problem now needs two letters, x and y, and two equations.
  Check: Two numbers add to 64, and one is 10 more. They are 27 and 37.
- **Ratios in proportion** (Class 8, Ganita Prakash Part 1, page 162).
  Here: the ratio test compares a_1/a_2 with b_1/b_2 in just this way.
  Check: $2 : 4$ and $3 : 6$ are both $1 : 2$, so they are in proportion.

- **Same step, both sides** (Class 8, Ganita Prakash Part 2, page 141).
  Here: elimination multiplies both sides by a number to match one letter.
  Check: $x + y = 5$ times 2 is $2x + 2y = 10$, and $(2, 3)$ fits both.

If any of these felt new, read the page named before going on.

One equation, one line, endless solutions

One line cannot pin a point, two can



Every point on the line solves $x + y = 5$, so one equation leaves endless answers. A second line meets it at one point, and that single point is what a pair of equations is for.

One line carries six dots as sample solutions, and a second line crossing it marks the single pair that fits both.

Let us take the equation $x + y = 5$. Try the pair $(1, 4)$. Substitute it in: $1 + 4 = 5$. It checks out, so $(1, 4)$ is a solution.

Now you try one. Check the pair $(2, 3)$ in the same equation. (Answer: $2 + 3 = 5$, so yes, it is a solution.)

Two more pairs work too: $(0, 5)$ and $(5, 0)$. A linear equation in x and y always has this same shape: $ax + by + c = 0$, with a and b not both zero.

Every pair (x, y) that satisfies it is a solution, and there are infinitely many such pairs. Plot every one of them, and they all land on the same straight line.

One equation on its own cannot pin down a single point. Finding one exact point takes a second equation, a second line, and that is what turns one equation into a pair.

Check that a and b are not both zero before calling an equation linear in x and y: if both are zero, there is no line left to talk about.

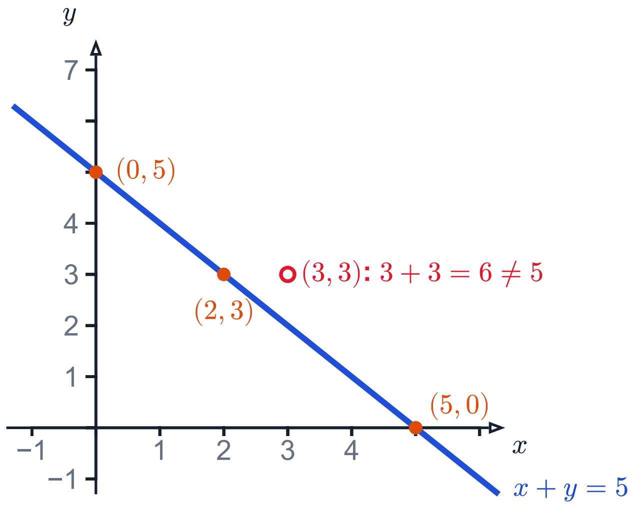
TRY
Is $(1, 4)$ a solution of $x + y = 5$?

Answer: Yes. $1 + 4 = 5$.

BACK
Where two lines meet · p. 69

RULE
Check that a and b are not both zero before calling an equation linear in x and y.

One equation, infinitely many solutions



The line for x plus y equals 5 passes through three marked points that satisfy it, and one nearby point that does not.

A SCHOLAR INDIA REMEMBERS

How many solutions does $x + y = 5$ have? Let us count a few. $(0, 5)$, $(1, 4)$, $(2, 3)$, $(2.5, 2.5)$. *We can keep going, with no end. Each pair is a point on the same line. So one equation alone never fixes x and y. We need a second equation.*

Aryabhata · the hoopoe

CHECK YOURSELF

3. The general form of a linear equation in x and y is $ax + by + c = 0$. What must be true of a and b?

- **A.** a and b must both be positive
- **B.** a and b must be equal
- **C.** a and b are not both zero
- **D.** a and b must both be whole numbers

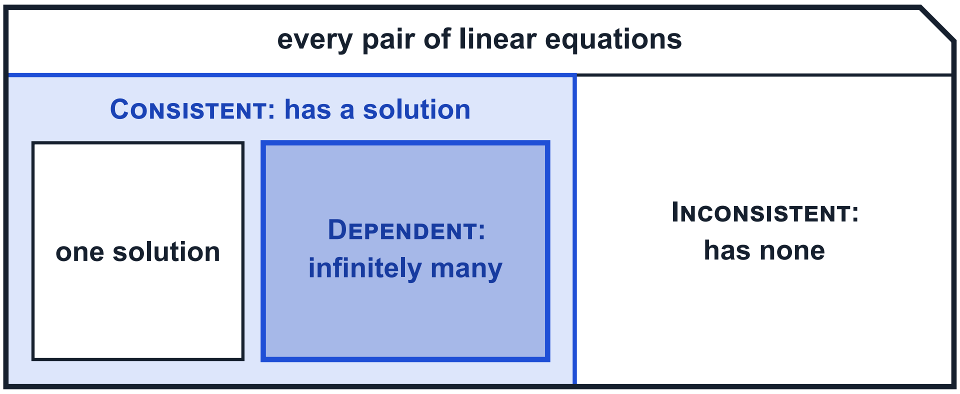
4. Why does a single linear equation such as $x + y = 5$ have infinitely many solutions (x, y)?

- **A.** because x and y can be swapped without changing the equation
- **B.** every point on the line it describes satisfies the equation
- **C.** because the equation has two variables written in it
- **D.** because 5 can be split into infinitely many pairs of factors

1 more in this set online.

Consistent, inconsistent, dependent

Dependent sits inside consistent



A pair is consistent when it has at least one solution and inconsistent when it has none. Dependent means infinitely many solutions, so a dependent pair is always consistent: it is a kind of consistent, not a third case beside it.

The three words are not three equal boxes: dependent is a room inside the consistent house, and only the inconsistent pairs stand outside it.

Let us go back to $x + y = 5$ and $x - y = 1$, the pair from the frame section. They share the point $(3, 2)$.

Check it in your own working: $3 + 2 = 5$ and $3 - 2 = 1$. Both hold, so $(3, 2)$ really is a shared solution.

A pair of linear equations with at least one shared solution like this is called CONSISTENT. A pair with no shared solution at all is INCONSISTENT.

A consistent pair can go further. Suppose every point on one line also lies on the other. Then there are infinitely many shared solutions, not just one, and the pair is also called DEPENDENT.

Try it yourself. Is a dependent pair also consistent, or is it a separate case? (Answer: dependent is always consistent, since sharing every point on the line means sharing at least one point too.)

CAUTION
Dependent is a KIND of consistent, not a third thing sitting apart from it.

KEY
Consistent = has a solution. Inconsistent = has none. Dependent = has infinitely many, and is always consistent.

CHECK YOURSELF

5. A pair of linear equations is called consistent when

- **A.** both equations look identical when written out
- **B.** it can be solved graphically
- **C.** the two equations have the same constant term
- **D.** it has at least one common solution

6. Why is a dependent pair of equations always also consistent?

- **A.** dependent pairs are graphed differently, so consistency does not apply to them
- **B.** dependency and consistency are two names for the same thing
- **C.** sharing infinitely many solutions is more than enough for consistency
- **D.** a dependent pair has no solution, and no solution counts as consistent

1 more in this set online.

Naming the three ratios of a pair

Let us take $3x + 2y - 12 = 0$ and $5x - 2y - 4 = 0$. Read the coefficients straight off: $a_1 = 3$ and $a_2 = 5$, so $a_1/a_2 = 3/5$.

Now find the other ratio yourself. Read off b_1 and b_2, and divide. (Answer: $b_1 = 2$ and $b_2 = -2$, so $b_1/b_2 = -1$.)

Put any pair into the form $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ first. Every pair then gives you the same three ratios to compare: a_1/a_2, b_1/b_2 and c_1/c_2.

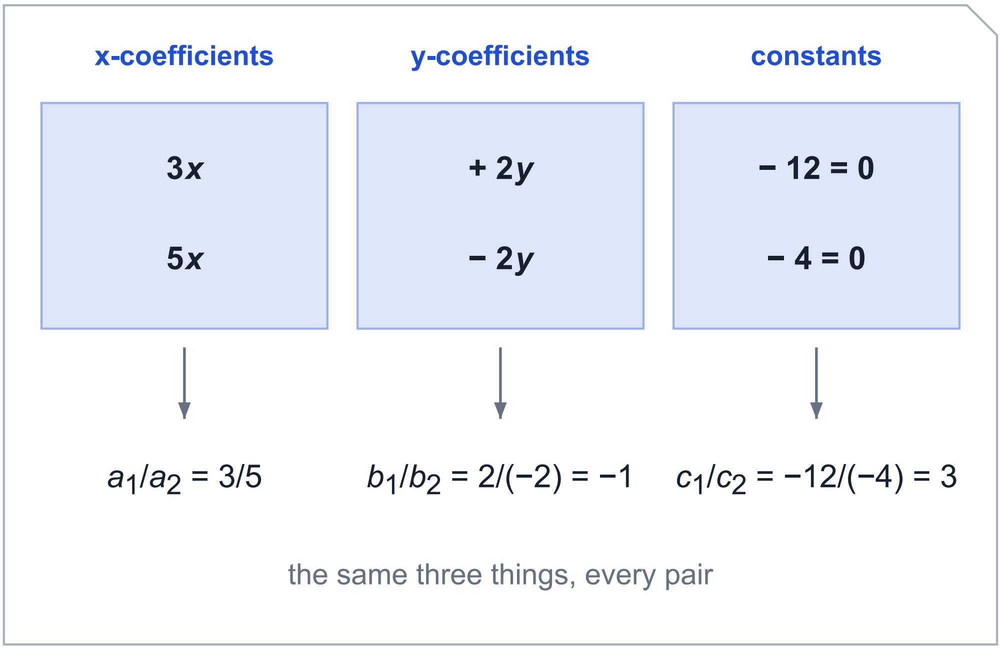
These three ratios compare the same three things every time: the x-coefficients, the y-coefficients, and the constant terms. Nothing else about the two equations matters for this comparison.

Write both equations in the form $ax + by + c = 0$ before you read off a single coefficient. A stray sign or a rearranged term throws every ratio off.

RULE
Write both equations in the form $ax + by + c = 0$ first. The ratios only work once the form matches.

KEY
Every pair compares the SAME three things: the ratio of the x-coefficients, the y-coefficients, and the constants.

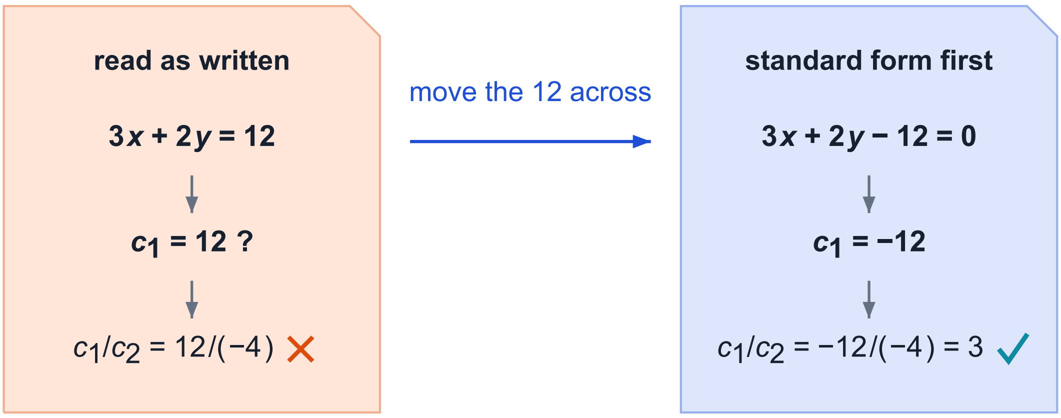
Three columns, three ratios



Line the two equations up in standard form and the three ratios read straight down the columns: x-coefficients, y-coefficients, constants. Here $a_1/a_2 = 3/5$ and $b_1/b_2 = -1$ already differ.

Two equations in standard form line up in three columns of ratios: x-coefficients, y-coefficients and constants.

A stray sign throws the third ratio off



Read the constant off $3x + 2y = 12$ and you take c_1 as 12; move it across to $3x + 2y - 12 = 0$ first and c_1 is -12. Only the second reading gives the ratio the test needs.

Writing $3x + 2y = 12$ in standard form first turns the third ratio's sign from wrong to c one over c two equals 3.

IN THE WILD

Ohm's law gives $V = IR$ for a resistor of resistance R. A real circuit element also carries a fixed extra voltage c, giving $V = IR + c$ instead, one linear equation in two unknowns, R and c. One reading cannot find both. A second reading at a different current gives a second such equation, and the two readings together are a pair of linear equations in two variables.

CHECK YOURSELF

7. For $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, which three ratios are compared to classify the pair?

- **A.** a_1/b_1, a_2/b_2, c_1/c_2
- **B.** a_1/a_2, b_1/b_2, c_1/c_2
- **C.** a_1/a_2 and b_1/b_2 only, with no constant ratio
- **D.** c_1/a_1, c_2/a_2, b_1/b_2

8. Why does the ratio test need the third ratio c_1/c_2, and not just a_1/a_2 and b_1/b_2?

- **A.** the third ratio decides which line is drawn first
- **B.** without it, the equations cannot be written in general form
- **C.** it is only needed when x and y both have coefficient 1
- **D.** it settles coincident versus merely parallel

1 more in this set online.

The ideas

The graphical method: plot, then read

Let us take $x + y = 5$. Set $x = 0$, and $y = 5$ follows. Set $y = 0$, and $x = 5$ follows. Two points, $(0, 5)$ and $(5, 0)$, and that is already enough to draw the line.

Now try the same trick on $x - y = 1$ yourself. Set each variable to zero in turn. (Answer: $(0, -1)$ and $(1, 0)$.)

This is the whole recipe for solving a pair graphically: find at least two solutions of each equation, plot them, and draw the line through each pair of points.

Two points are enough to draw a straight line, so pick easy ones for your own working. Often the easiest points are where the line crosses each axis, since one coordinate is then simply zero.

Once both lines are drawn, the answer sits in how they meet. Every method for solving such a pair finds the one point that lies on both lines, if there is one; this one just finds it by drawing. A single crossing point is that point, and the pair's unique solution. Parallel lines that never meet mean no solution. Lines that coincide give every point on the line as a solution.

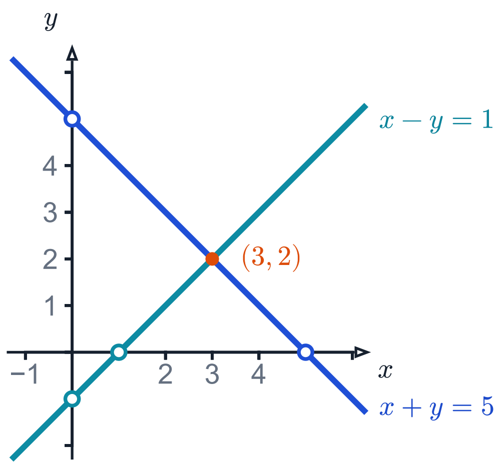
A solution read off a graph is only as exact as the plotting. Coordinates that are not whole numbers are easy to misread, which is exactly why the two algebraic methods ahead matter.

BACK
One equation, one line, endless solutions · p. 72

CAUTION
A solution read off a graph is only as exact as the plotting. Coordinates that are not whole numbers are easy to misread.

KEY
Two points are enough to draw a line. Pick easy ones, such as where the line crosses each axis.

Two points, a line, and where it crosses



Two lines are each drawn from two points, and they cross at the pair's one solution.

CHECK YOURSELF

9. To draw the line for one linear equation in x and y, how many solutions of that equation must be plotted?

- **A.** exactly one, since a single point already lies on the line
- **B.** at least three, to be safe
- **C.** at least two, since two points fix a straight line
- **D.** as many as the equation's coefficients

10. After both lines are drawn, how is the number of solutions read off the graph?

- **A.** by whether the lines cross once, never, or coincide
- **B.** by measuring the angle between the two lines in degrees
- **C.** by comparing which line is drawn higher on the page
- **D.** by counting how many points were plotted in total

1 more in this set online.

Unequal first two ratios: one solution

Let us go back to $3x + 2y - 12 = 0$ and $5x - 2y - 4 = 0$ from the last section. There, $a_1/a_2 = 3/5$ and $b_1/b_2 = -1$.

Check it yourself: is $3/5$ equal to -1? (Answer: no, $3/5 \neq -1$.)

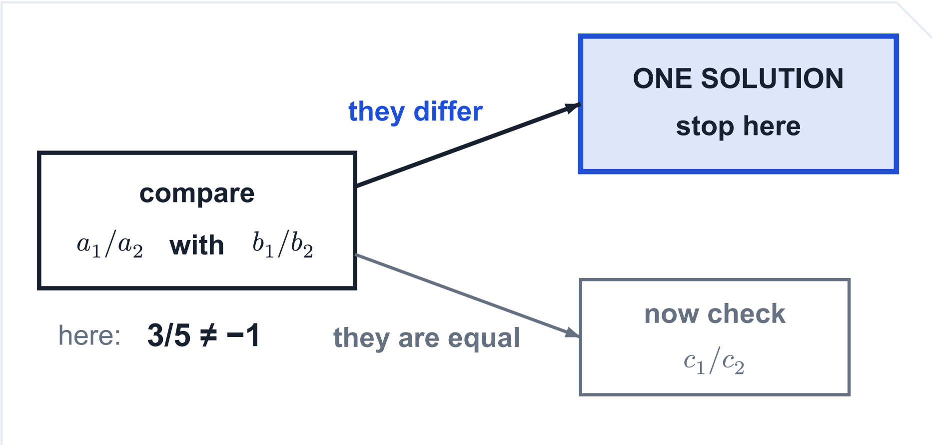
Whenever $a_1/a_2 \neq b_1/b_2$ like this, the two lines have different slopes. Different slopes can only cross once: two straight lines with different slopes are never parallel, so they meet at exactly one point.

This is the easiest of the three cases to spot in your own work. Check the first two ratios only, and if they differ, the answer is already settled. You never need to check the third ratio c_1/c_2 at all.

BACK
Naming the three ratios of a pair · p. 75

KEY
Different slopes always cross exactly once.

Two ratios differ, and you can stop



Compare a_1/a_2 with b_1/b_2 first. If they differ, the slopes differ, the lines cross exactly once, and the third ratio is never needed. Only equal first ratios send you on to c_1/c_2.

A decision path compares a1 over a2 with b1 over b2, checking c1 over c2 when those two agree.

CHECK YOURSELF

11. The ratio test says a pair of linear equations has a unique solution when

- **A.** $a_1/a_2 = b_1/b_2$
- **B.** $a_1/a_2 \neq c_1/c_2$
- **C.** $a_1 \neq a_2$ and $b_1 \neq b_2$
- **D.** $a_1/a_2 \neq b_1/b_2$

12. Why does $a_1/a_2 \neq b_1/b_2$ guarantee the two lines intersect at exactly one point?

- **A.** it shows the lines pass through the origin
- **B.** it shows both equations have the same constant term
- **C.** it shows the lines have different slopes
- **D.** it shows one line is shorter than the other

2 more in this set online.

First two equal, third not: no solution

Let us take pair 2 from the same comparison: $6x + 3y - 15 = 0$ and $2x + y - 6 = 0$. Here $a_1/a_2 = b_1/b_2 = 3$.

Now find the third ratio yourself, in your own calculation. (Answer: $c_1/c_2 = 5/2$, and $5/2$ is not 3.)

Whenever $a_1/a_2 = b_1/b_2 \neq c_1/c_2$ like this, the pair has no solution. The first two ratios being equal means the two lines share the same slope. The third ratio failing to match means they are not the same line.

A pair like this is inconsistent. There is no point that satisfies both equations at once, because the lines never touch.

KEY
Same slope, different line: parallel, and never meeting.

CHECK YOURSELF

13. The ratio test says a pair of linear equations has no solution when

- **A.** $a_1/a_2 = b_1/b_2 = c_1/c_2$, the dependent case
- **B.** $a_1/a_2 \neq b_1/b_2$
- **C.** $a_1/a_2 = b_1/b_2 \neq c_1/c_2$
- **D.** $a_1/a_2 \neq c_1/c_2$

14. Why does $a_1/a_2 = b_1/b_2 \neq c_1/c_2$ mean the lines are parallel but not the same line?

- **A.** same slope from a, b; different position from c
- **B.** an unequal c-ratio means the lines have different slopes
- **C.** equal a- and b-ratios already prove the lines are identical
- **D.** the condition means one equation is wrong

2 more in this set online.

All three ratios equal: infinite solutions

Let us take pair 3 from the same comparison: $4x + 6y - 10 = 0$ and $2x + 3y - 5 = 0$.

Check all three ratios yourself. (Answer: $a_1/a_2 = 4/2 = 2$, $b_1/b_2 = 6/3 = 2$, and $c_1/c_2 = 10/5 = 2$. All three are equal.)

Whenever $a_1/a_2 = b_1/b_2 = c_1/c_2$ like this, the pair has infinitely many solutions. All three ratios matching means the two equations describe the exact same line, just scaled by a constant factor.

This pair is dependent, and a dependent pair is always consistent. Every point on the shared line is a valid solution for you to check.

KEY
All three ratios equal means the two equations are the same line in disguise.

CHECK YOURSELF

15. The ratio test says a pair of linear equations has infinitely many solutions when

- **A.** $a_1/a_2 = b_1/b_2 \neq c_1/c_2$
- **B.** $a_1/a_2 = b_1/b_2 = c_1/c_2$
- **C.** $a_1/a_2 \neq b_1/b_2 = c_1/c_2$
- **D.** $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$

16. Why does $a_1/a_2 = b_1/b_2 = c_1/c_2$ mean the two equations describe the same line?

- **A.** it means both equations have the same number of terms
- **B.** it means the two lines are parallel but placed side by side
- **C.** it means x and y have swapped roles between the equations
- **D.** one equation is a constant multiple of the other

2 more in this set online.

The substitution method, step by step

Let us take $x + y = 10$ and $2x - y = 5$. From the first equation, $x = 10 - y$.

The substitution method always runs in three steps:

- **Step one:** write one variable in terms of the other, from the equation where that is easiest. Here, $x = 10 - y$.
- **Step two:** substitute that expression into the **other** equation. Here, that means putting $10 - y$ in place of x in $2x - y = 5$. Two variables become one, and one equation in one variable is something you already know how to solve.
- **Step three:** substitute the value you find back into either equation, to get the remaining variable.

Try all three steps yourself, before you read the full working in the next worked example. (Answer: the pair solves to $x = 5$ and $y = 5$.)

Solve for the variable with the simplest coefficient. It keeps the numbers small.

Watch what the final equation looks like: a statement that is always true, like $0 = 0$, means infinitely many solutions, and a statement that is always false, like $0 = 9$, means no solution.

CAUTION
A true statement with no variable left, like $0 = 0$, means infinitely many solutions, not “no work needed”.

RULE
Solve for the variable with the simplest coefficient. That keeps the substitution step arithmetic light.

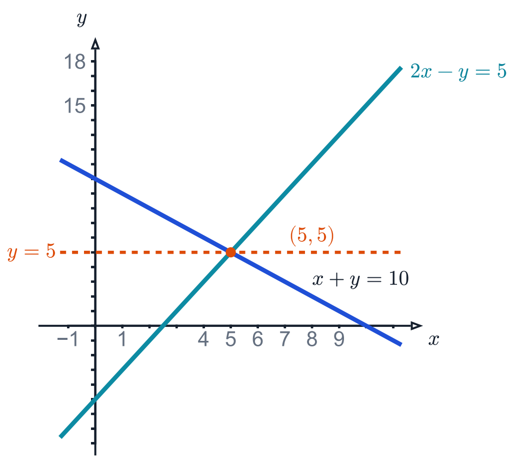
BACK
The graphical method: plot, then read · p. 77

A SCHOLAR INDIA REMEMBERS

You solved one pair by substitution. Will the same steps solve the next pair? Write the steps so that someone else could follow them. First, make one unknown the subject of one equation. Then put it into the other equation. Solve that. Then go back and find the other unknown.

Aryabhata · the hoopoe

Substitution’s answer is also a line



The two original lines cross the extra line, y equals 5, at the same point substitution’s last step finds.

CHECK YOURSELF

17. The substitution method solves a pair of linear equations by

- **A.** multiplying both equations until one variable's coefficients match
- **B.** plotting both equations and reading off the crossing point
- **C.** solve for one variable, then substitute it in
- **D.** guessing values of x and y until both equations hold

18. Why does substituting one variable's expression into the other equation make the problem solvable?

- **A.** it removes the constant term from the equation
- **B.** it leaves one equation, one unknown
- **C.** it makes both equations identical
- **D.** it changes the equation from linear to quadratic

2 more in this set online.

The elimination method, step by step

Let us take $2x + 3y = 13$ and $3x + 2y = 12$. Multiply the first equation by 2 and the second by 3.

Check the y-coefficient in each yourself. (Answer: both become $6y$.)

That is the elimination method's first step: multiply one or both equations by constants, so one variable ends up with equal coefficients in both.

Second, add or subtract the two equations to eliminate that variable, leaving one equation in one variable to solve. Elimination, like the other methods here, finds the one point that lies on both lines; it just gets there by cancelling a variable instead of plotting or substituting.

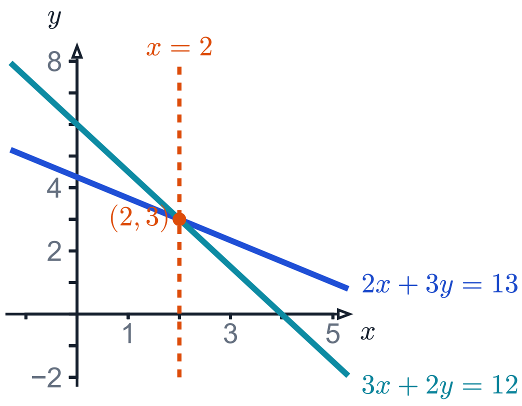
Same sign on the matched coefficients: subtract. Opposite signs: add. Getting this backwards doubles the variable instead of removing it, so check the signs on your own pair before doing either.

As with substitution, an always-true statement left at the end means infinitely many solutions. An always-false one means no solution.

RULE
Same sign on the matched coefficients: subtract. Opposite signs: add.

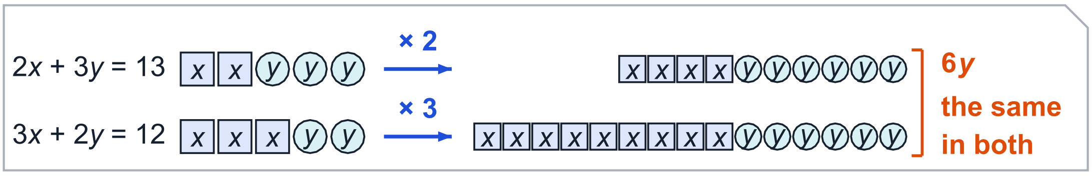
KEY
Make one variable's coefficients match in size, then add or subtract to remove it.

Elimination's answer is also a line



The two original lines cross the extra line, x equals 2, at the same point elimination's subtraction step finds.

Match the y-chips, then they cancel



match the y-chips, then subtract and they cancel

Multiplying by 2 and by 3 makes both equations carry six y-chips.
Subtract one row from the other and the y-chips cancel, leaving
x-chips alone: one equation, one unknown.

Multiplying each equation by the other's y-coefficient is what makes the y-chips match, and matched chips are the ones a subtraction can remove.

A SCHOLAR INDIA REMEMBERS

In 499 CE, Aryabhata gave a step-by-step method for a whole class of equations. It was not a trick for one equation. The same steps worked for every equation in the class. Elimination is a method of that kind. It works the same way on every pair of linear equations in two unknowns. Learn the steps once, and they work for the next pair too.

Aryabhata · the hoopoe

CHECK YOURSELF

19. The elimination method solves a pair of linear equations by

- **A.** writing one variable in terms of the other and substituting it in
- **B.** matching a coefficient, then adding or subtracting
- **C.** plotting both lines and reading off where they cross
- **D.** dividing both equations by their constant terms

20. Why multiply the equations by constants before adding or subtracting them?

- **A.** to make one variable's coefficient equal in both
- **B.** to make both equations have the same constant term
- **C.** to make the equations easier to plot on a graph
- **D.** to remove any variable that has a negative coefficient

2 more in this set online.

Turning words into a pair of equations

Let us take a problem: two numbers add to 45, and differ by 5.

Name the two unknown quantities first, in words, before you write a single equation. Here, let x and y be the two numbers.

Now translate each condition yourself, one at a time, into one equation. (Answer: $x + y = 45$ and $x - y = 5$.)

That is the first of the two steps a situational problem always takes. Second, solve the resulting pair by substitution or elimination.

A solved x and y that satisfy both equations can still fail the original story. Check them against the words, not just against the algebra: an answer can solve the equations correctly and still make no sense as an age, a distance, or a cost.

BACK
The substitution method, step by step · p. 81

AHEAD
Turning a situation into a quadratic · p. 116

CAUTION
A solved x and y that satisfy both equations can still fail the original story. Check them against the words, not just the algebra.

A SCHOLAR INDIA REMEMBERS

A word problem asks for two unknown numbers. Before you write any equation, name them. Let x be the cost of one pen and y the cost of one notebook, in rupees. Then find two facts in the problem that link x and y. Each fact gives one equation. Two facts give the pair you need.

Aryabhata · the hoopoe

CHECK YOURSELF

21. Turning a word problem into a pair of linear equations involves

- **A.** guessing two numbers that sound reasonable for the problem
- **B.** writing one very long equation that contains every condition at once
- **C.** solving the problem by trial and error before writing any equation
- **D.** naming the unknowns, then writing each condition as an equation

22. Why check the values of x and y against the original word problem after solving?

- **A.** to make the handwriting neater before submitting
- **B.** because substitution and elimination sometimes give the wrong answer
- **C.** to confirm the answer fits the situation described
- **D.** to round the answer to the nearest whole number

1 more in this set online.

Common mistakes

No solution is not the same as infinite

No solution and infinitely many solutions can feel like the same kind of failure. Neither one hands you a single tidy answer.

They are opposite outcomes, and the third ratio is the only thing that tells them apart.

Let us take $2x + 4y - 6 = 0$ and $x + 2y - 5 = 0$. The first two ratios match: $a_1/a_2 = b_1/b_2 = 2$. But the third does not: $c_1/c_2 = 6/5$. Same slope, different line: parallel, and no solution.

Now change only the second equation's constant term yourself, from -5 to -3. (Answer: $x + 2y - 3 = 0$, and the third ratio becomes $c_1/c_2 = 6/3 = 2$, which now matches the other two.)

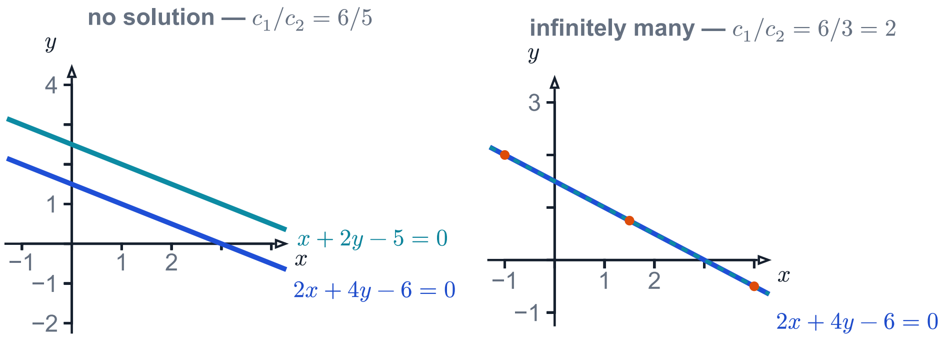
The same pair of lines is coincident instead of parallel: infinitely many solutions, not none.

BACK
First two equal, third not: no solution · p. 79

CAUTION
No solution and infinitely many solutions are opposite outcomes, not two kinds of the same failure.

KEY
The third ratio is what tells parallel apart from coincident. Never skip comparing it.

The third ratio flips the outcome



Right panel: the second equation is drawn dashed directly over the first — one constant term changed, and the same line now carries two names instead of one.

The same first line is paired with two different lines: one pair stays parallel, and the other pair lands exactly on top of it.

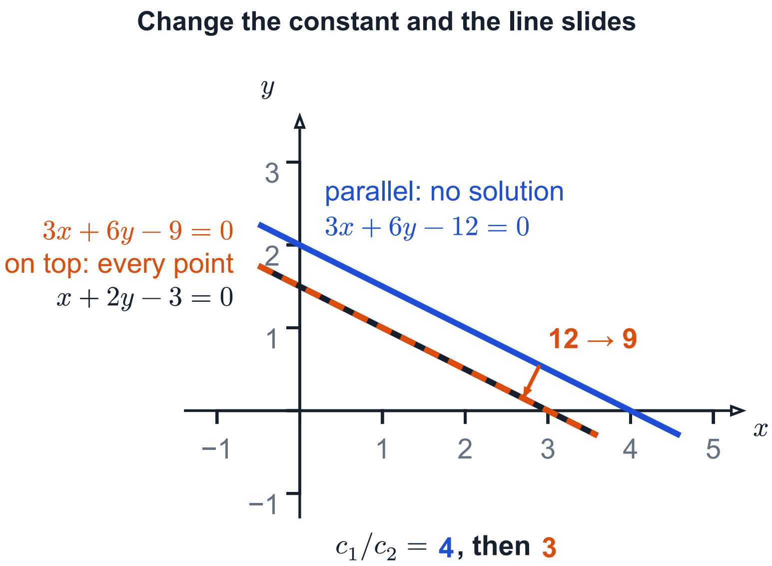
DOES A PAIR WHOSE FIRST TWO RATIOS MATCH HAVE NO SOLUTION, OR INFINITELY MANY?

WEAK
A student checks the first two ratios: $a_1/a_2 = 3$ and $b_1/b_2 = 3$. Both match, so the student stops there and says the pair has infinitely many solutions.

That is wrong. The third ratio is $c_1/c_2 = 12/3 = 4$, and 4 is not 3. Two matching ratios are never enough on their own.

STRONG
We check all three ratios before deciding. $a_1/a_2 = 3$, $b_1/b_2 = 3$, and $c_1/c_2 = 12/3 = 4$. The first two match, the third does not, so the pair has no solution: the lines are parallel.

Change only the constant term, from 12 to 9: $3x + 6y - 9 = 0$ paired with the same $x + 2y - 3 = 0$. Now $c_1/c_2 = 9/3 = 3$, matching the other two. The same two lines are now one line, and the pair has infinitely many solutions.



With 12 the second line runs beside the first and never meets it; with 9 it lies on the first line and meets it everywhere. Only the constant moved, and only the third ratio noticed.

The first two ratios stay equal while only the constant changes, and that one change is what moves the line from parallel to coincident.

WATCH OUT

The trap: Seeing $a_1/a_2 = b_1/b_2$ match, the student stops and never checks c_1/c_2 at all.

The correction: Matching the third ratio means one line drawn twice, so every point on it solves both equations; a third ratio that differs means parallel lines, so no point solves both.

CHECK YOURSELF

23. A student says: "no solution and infinitely many solutions are basically the same kind of failure, since the equations just do not give one clean answer." What is wrong with this?

- **A.** nothing is wrong; both cases mean the equations cannot be solved
- **B.** they are opposite, told apart by c_1/c_2
- **C.** the difference is only about how many times you try substitution
- **D.** no solution and infinitely many are the same whenever $a_1/a_2 = b_1/b_2$

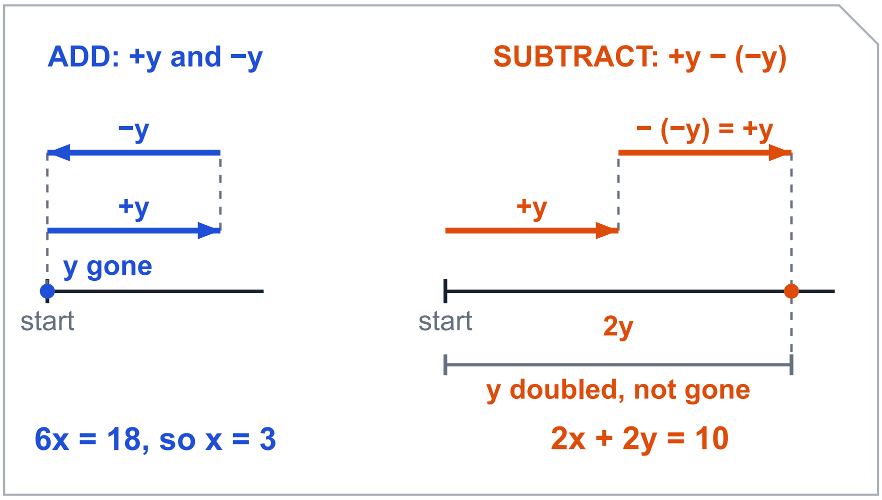
24. For $2x + 4y - 6 = 0$ and $x + 2y - 5 = 0$, a student sees $a_1/a_2 = b_1/b_2 = 2$ and concludes the pair has infinitely many solutions. What went wrong?

- **A.** nothing went wrong; two matching ratios are already enough
- **B.** the student should use $c_1 = 6$ and $c_2 = 5$ directly, not their ratio
- **C.** the mistake is in computing a_1/a_2, which should be 1
- **D.** $c_1/c_2 = 6/5$, which is not 2, so the pair actually has no solution

3 more in this set online.

Same signs subtract, opposite signs add

Opposite signs add to nothing; subtracting doubles



The y-coefficients are equal and opposite. Add the equations and the two y-arrows cancel, leaving 6x = 18, so x = 3. Subtract and the second arrow flips, so y is doubled to 2y instead of removed.

Adding the two equations cancels the y-terms to give 6x equals 18, but subtracting doubles them to 2x plus 2y equals 10.

Matching coefficients does not always mean subtract.

Let us take $4x + y = 14$ and $2x - y = 4$. The y-coefficients are already matched in size, at $+1$ and -1.

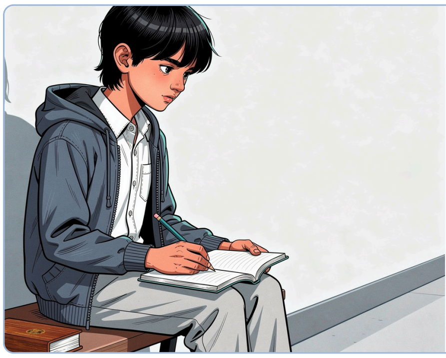
Try subtracting the equations yourself. (Answer: $(4x + y) - (2x - y) = 14 - 4$ gives $2x + 2y = 10$, and y has not been eliminated.)

The two coefficients have opposite signs, so subtracting doubles y instead of cancelling it. Adding is what works here: $(4x + y) + (2x - y) = 14 + 4$ gives $6x = 18$, so $x = 3$. Substituting back, $y = 14 - 4(3) = 2$.

BACK
The elimination method, step by step · p. 82

CAUTION
Read the signs of the two matched terms before choosing. Same sign, subtract; opposite signs, add.

KEY
If the variable is still there after the step, the wrong operation was chosen.



FIRST TRY
~~The y-terms in $4x + y = 14$ and $2x - y = 4$ are matched in size, so I subtracted and got $2x + 2y = 10$.~~

SECOND LOOK
Subtracting cancels a term only when the matched coefficients share a sign. Here they are $+1$ and -1, opposite signs, so adding cancels them. That gives $6x = 18$, so $x = 3$ and $y = 2$.

Matching size is not enough. Same sign, subtract. Opposite sign, add.

THE COEFFICIENTS OF ONE VARIABLE ALREADY MATCH IN SIZE. SUBTRACT, OR ADD?

WEAK
A student sees $+2y$ and $-2y$, both matched in size, and subtracts the equations because that is what worked last time. $(5x+2y)-(3x-2y)=19-5$ gives $2x+4y=14$.

The y term has not gone anywhere. Subtracting two terms with opposite signs doubles them instead of cancelling them.

STRONG
We check the signs first. The y-coefficients are $+2$ and -2: opposite signs, so we add, not subtract. $(5x+2y)+(3x-2y)=19+5$ gives $8x=24$, so $x=3$.

Substituting back, $3(3)-2y=5$, so $2y=4$ and $y=2$. Check: $5(3)+2(2)=19$ and $3(3)-2(2)=5$. Both hold.

Worked examples

WORKED EXAMPLE · **Solving x + y = 5 and x - y = 1 by graphing**

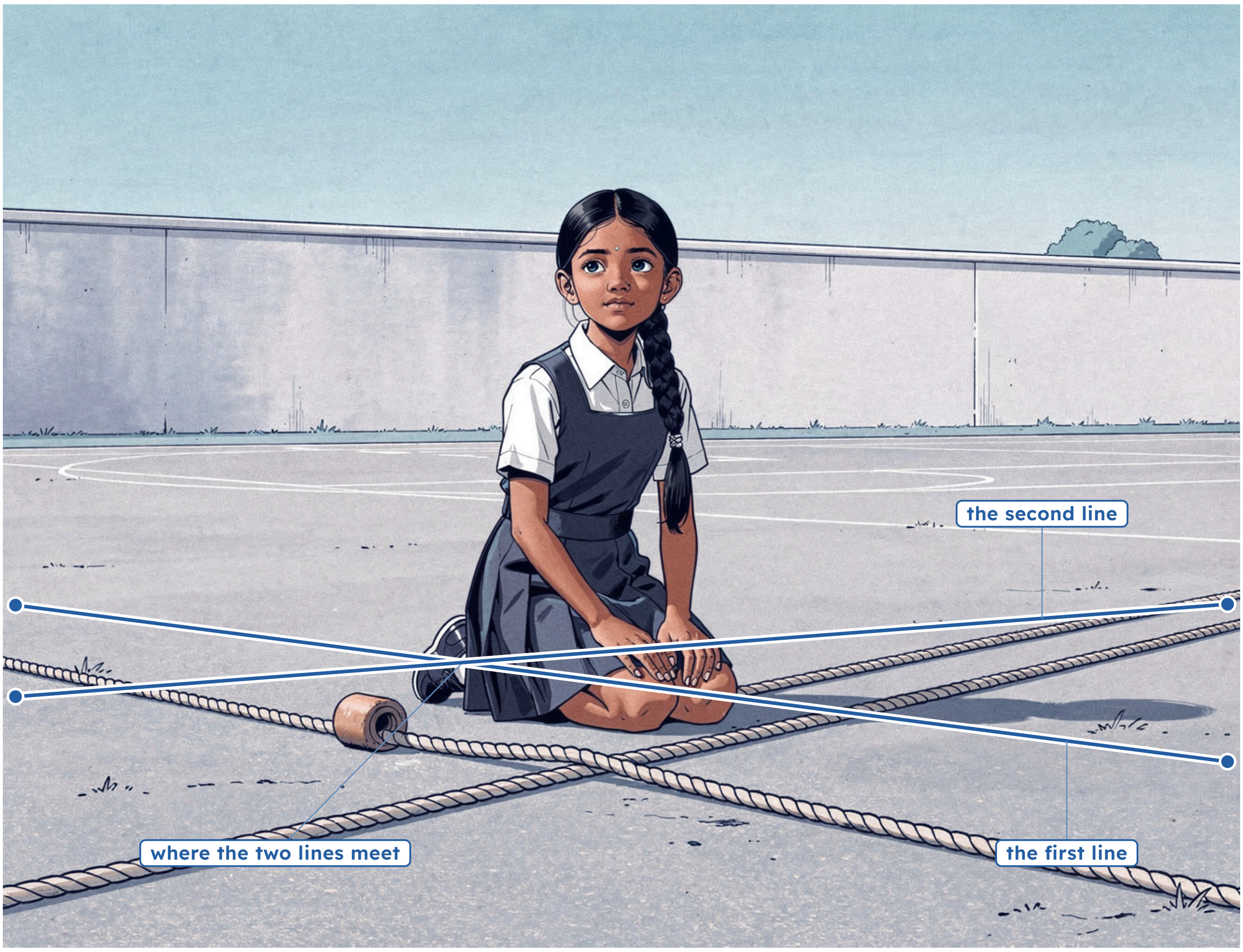
1.	$x+y=5$ gives $(0,5)$ and $(5,0)$	*Set $x=0$, and $y=5$. Set $y=0$, and $x=5$. Two points are enough to draw the line.*
2.	$x-y=1$ gives $(0,-1)$ and $(1,0)$	*Now you try the same trick on the second equation, setting each variable to zero in turn.*
3.	the two lines cross at $(3,2)$	*Plot both pairs of points, draw each line, and read off where they meet.*
4.	$3+2=5$ and $3-2=1$	CHECK ***Check the crossing point by substituting it back into both original equations.*** *Both hold, so $(3,2)$ really is the pair's solution.*

TRY
For $x+y=4$ and $x-y=2$, where do the two lines cross?

Answer: $(3,1)$, confirmed by substitution: $3+1=4$, $3-1=2$.

RULE
After reading the crossing point off the graph, confirm it by substituting into both original equations.

BACK
The graphical method: plot, then read · p. 77



Nila kneels where two straight ropes laid across the yard cross at a single point.

SOLVE ANY PAIR OF LINEAR EQUATIONS BY GRAPHING.

1. **Find two points on each line** Set $x = 0$ to get one point, then set $y = 0$ to get another. For $x + y = 5$, that gives $(0, 5)$ and $(5, 0)$.
2. **Plot and join each pair** Mark both points for one equation, and draw the line through them. Do the same for the second equation, $x - y = 1$, using $(0, -1)$ and $(1, 0)$.
3. **Read the crossing point** Look at where the two lines meet. Here, they cross at $(3, 2)$.
4. **Check by substituting** Put $(3, 2)$ back into both equations. $3 + 2 = 5$ and $3 - 2 = 1$, so it checks out.
5. **Name the case** One crossing point means one solution. Parallel lines mean no solution. The same line twice means infinitely many.

CHECK YOURSELF

25. Solving $x + y = 5$ and $x - y = 1$ graphically, the worked example plots $(0, 5)$ and $(5, 0)$ for the first line. Where do these points come from?

- **A.** they are the midpoint and endpoint of the eventual crossing point
- **B.** they come from solving both equations together
- **C.** they are chosen at random from any pair of numbers
- **D.** setting $x = 0$ gives $y = 5$, and setting $y = 0$ gives $x = 5$

26. The two lines in the worked example cross at $(3, 2)$. What confirms this is the pair's solution?

- **A.** $(3, 2)$ lies exactly halfway between the four plotted points
- **B.** $3 + 2 = 5$ and $3 - 2 = 1$: it satisfies both
- **C.** 3 and 2 are both smaller than 5
- **D.** the point was read off the graph, so it needs no further check

1 more in this set online.

Three pairs, nine ratios, three verdicts

	a_1/a_2	b_1/b_2	c_1/c_2	verdict
Pair 1	3/5	−1	not needed	one solution
Pair 2	3	3	5/2	no solution
Pair 3	2	2	2	infinitely many

Read the first two ratios of each pair. If they differ the pair has one solution and the third ratio is never looked at; if they match, the third ratio decides between no solution and infinitely many.

Nine ratios were possible but only eight were computed: the ratio test reads left to right and stops the moment the verdict is settled.

WORKED EXAMPLE · Classifying three pairs by ratio, with no solving or graphing

1.	$3x + 2y - 12 = 0$ and $5x - 2y - 4 = 0$: $a_1/a_2 = 3/5$, $b_1/b_2 = -1$	*Read the three coefficients straight off each equation.*
2.	$3/5 \neq -1$, so the pair has a unique solution and is consistent	*The first two ratios already differ, so there is no need to check the third ratio at all.*
3.	$6x + 3y - 15 = 0$ and $2x + y - 6 = 0$: $a_1/a_2 = b_1/b_2 = 3$	*The first two ratios match this time.*
4.	$c_1/c_2 = 5/2$	*Check the third ratio too, since the first two matched.*
5.	$3 \neq 5/2$, so the pair has no solution	*Same slope, different line: the lines are parallel.*
6.	$4x + 6y - 10 = 0$ and $2x + 3y - 5 = 0$: $a_1/a_2 = b_1/b_2 = c_1/c_2 = 2$	*All three ratios come out equal.*
7.	the pair has infinitely many solutions	*All three ratios matching means the two equations describe the same line.*
8.	$4(1) + 6(1) - 10 = 0$ and $2(1) + 3(1) - 5 = 0$	CHECK ***Check pair 3 yourself by picking any point and testing it in both equations.*** *Try $x = 1$, $y = 1$: it satisfies both, confirming the two lines really are the same one.*

BACK
Unequal first two ratios: one solution · p. 78

TRY
Classify $2x + 3y - 7 = 0$ and $4x + 6y - 9 = 0$ without solving.

Answer: $a_1/a_2 = b_1/b_2 = 1/2$, $c_1/c_2 = 7/9$. Parallel, no solution.

CAUTION
Keep the sign in every ratio: $b_1/b_2 = 2/(-2) = -1$ here, not 1. Dropping it misclassifies the pair.

CLASSIFY ANY PAIR OF LINEAR EQUATIONS BY RATIO, WITHOUT SOLVING OR GRAPHING.

1. **Write both equations in standard form** Put each equation as $ax + by + c = 0$ before reading off a single coefficient.
2. **Compare the first two ratios** Read off a_1/a_2 and b_1/b_2. If they differ, the pair has a unique solution, and you can stop here.
3. **Check the third ratio if needed** If $a_1/a_2 = b_1/b_2$, also find c_1/c_2. A match with the first two means infinitely many solutions; a mismatch means no solution.
4. **Match the case to the example** $3x + 2y - 12 = 0$ and $5x - 2y - 4 = 0$ give $3/5 \neq -1$: one solution. $6x + 3y - 15 = 0$ and $2x + y - 6 = 0$ give equal first ratios but $3 \neq 5/2$: no solution. $4x + 6y - 10 = 0$ and $2x + 3y - 5 = 0$ give all three ratios equal to 2: infinitely many.

CHECK YOURSELF

27. In the worked example, Pair 1 is $3x + 2y - 12 = 0$ and $5x - 2y - 4 = 0$, classified as a unique solution from $3/5 \neq -1$. Which ratios were actually compared to reach this?

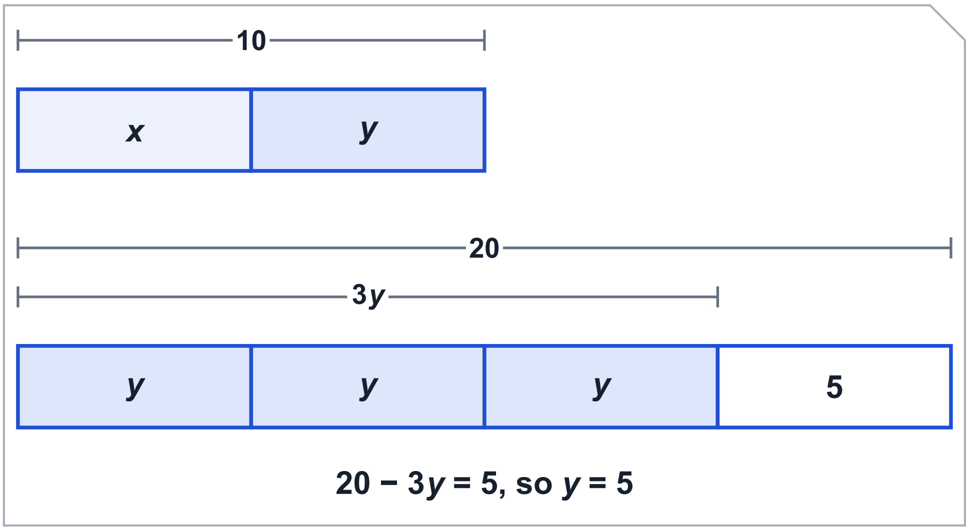
- **A.** $a_1/a_2 = 3/5$ against $c_1/c_2 = 3$
- **B.** $a_1/a_2 = 3/5$ against $b_1/b_2 = -1$
- **C.** $b_1/b_2 = -1$ against $c_1/c_2 = 3$
- **D.** $a_1/a_2 = 3/5$ against itself

28. The worked example's Pair 2, $6x + 3y - 15 = 0$ and $2x + y - 6 = 0$, is classified as no solution. What makes it different from Pair 3, which has infinitely many solutions?

- **A.** Pair 2's constant ratio breaks the match; Pair 3's does not
- **B.** Pair 2's a- and b-ratios are not equal to each other, unlike Pair 3's
- **C.** Pair 2 has larger coefficients than Pair 3
- **D.** Pair 2 was solved by substitution and Pair 3 by elimination

1 more in this set online.

Twenty, less three y's, leaves five



Putting x = 10 − y into the second equation leaves one bar: 20 with three y-blocks and a 5. Three equal blocks share what is left, so each y is 5, and x = 10 − 5 = 5.

Two bars show 20 minus 3y equals 5, so y equals 5, once x is written as 10 minus y.

WORKED EXAMPLE · **Solving x + y = 10 and 2x - y = 5 by substitution**

1.	$x = 10 - y$	*We write x in terms of y, from the first equation.*
2.	$2(10 - y) - y = 5$	*Now substitute that into the second equation yourself.*
3.	$20 - 3y = 5$, so $y = 5$	*Simplify, then solve the single remaining variable.*
4.	$x = 10 - 5 = 5$	*Substitute $y = 5$ back into $x = 10 - y$.*
5.	$x = 5, y = 5$ satisfies both equations	*Checking the solution against both original equations confirms it.*
6.	$5 + 5 = 10$ and $2(5) - 5 = 5$	CHECK ***Check by substituting $x = 5, y = 5$ back into both original equations.*** *Both hold, confirming the solution.*

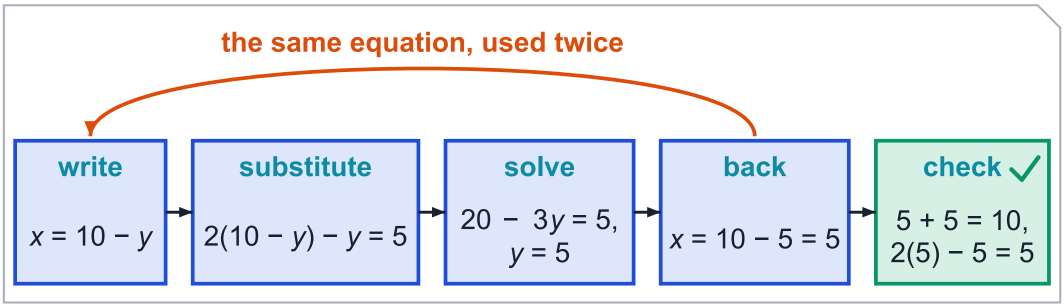
BACK
The substitution method, step by step · p. 81

TRY
Solve $x + y = 7$ and $3x - y = 5$ by substitution.

Answer: $x = 3, y = 4$.

CAUTION
Distribute the coefficient across the whole bracket: $2(10 - y) = 20 - 2y$, not $20 - y$.

Five steps and one check



Substitution is one equation used twice: first to write x in terms of y, then, after y is found, to write x as a number. The check at the end is what makes the answer a solution and not a guess.

Five boxes carry the substitution method from writing x equals 10 minus y through to the checked answer, 5 and 5.

SOLVE ANY PAIR OF LINEAR EQUATIONS BY SUBSTITUTION.

1. **Write one variable in terms of the other** Pick the equation and the variable that gives the simplest expression. From $x + y = 10$, that is $x = 10 - y$.
2. **Substitute into the other equation** Put $10 - y$ in place of x in $2x - y = 5$. That leaves one equation in one variable, y.
3. **Solve for that variable** Simplify $2(10 - y) - y = 5$ to $20 - 3y = 5$, so $y = 5$.
4. **Substitute back** Put $y = 5$ into $x = 10 - y$ to get $x = 5$.
5. **Check both equations** $5 + 5 = 10$ and $2(5) - 5 = 5$. Both hold, so $x = 5$, $y = 5$ is the solution.

CHECK YOURSELF

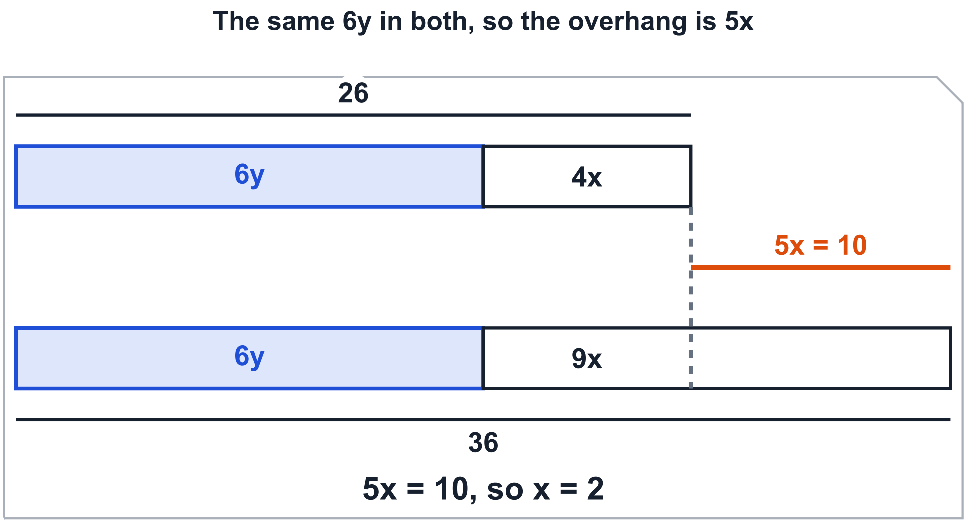
29. Solving $x + y = 10$ and $2x - y = 5$ by substitution, the worked example starts from $x = 10 - y$. Which equation does this come from?

- **A.** $2x - y = 5$, rearranged to isolate x
- **B.** $x + y = 10$, rearranged to isolate x
- **C.** both equations added together
- **D.** neither equation; it is assumed as a starting guess

30. Substituting $x = 10 - y$ into $2x - y = 5$, a student writes $2(10 - y) - y = 5$ and simplifies to $20 - y = 5$. What went wrong?

- **A.** the substitution itself is wrong; $x = 10 - y$ should not be used here
- **B.** the final answer $y = 5$ is wrong even though the working shown is correct
- **C.** $-y$ should have been added to 20, not subtracted
- **D.** $2(10 - y)$ expands to $20 - 2y$, so the equation should read $20 - 3y = 5$

1 more in this set online.



Scaled so both carry the same 6y-block, the bars lie side by side and the longer one sticks out by 36 − 26 = 10, which is the extra 5x: x = 2.

Elimination is a subtraction of bars: match one block in both, and whatever the longer bar overhangs by is the other unknown on its own.

WORKED EXAMPLE · Solving 2x + 3y = 13 and 3x + 2y = 12 by elimination

1.	$4x + 6y = 26$ and $9x + 6y = 36$	*We multiply the first equation by 2 and the second by 3, so y has equal coefficients in both.*
2.	$5x = 10$, so $x = 2$	*Subtract the two equations to eliminate y, then solve for x yourself.*
3.	$2(2) + 3y = 13$, so $y = 3$	*Substitute $x = 2$ back into the first original equation.*
4.	$x = 2$, $y = 3$ satisfies both equations	*Checking against both original equations confirms the solution.*
5.	$2(2) + 3(3) = 4 + 9 = 13$ and $3(2) + 2(3) = 6 + 6 = 12$	CHECK ***Check by substituting $x = 2, y = 3$ back into both original equations.*** *Both hold, confirming the solution.*

BACK
The elimination method, step by step · p. 82

TRY
Solve $3x + 2y = 12$ and $x + 2y = 8$ by elimination.

Answer: $x = 2, y = 3$.

CAUTION
Subtract in the same order on both sides, or the sign of x's coefficient comes out wrong.

SOLVE ANY PAIR OF LINEAR EQUATIONS BY ELIMINATION.

1. Match one variable's coefficients Multiply one or both equations by constants. For $2x + 3y = 13$ and $3x + 2y = 12$, multiplying by 2 and 3 gives $4x + 6y = 26$ and $9x + 6y = 36$.

2. Check the signs, then add or subtract Matched coefficients with the same sign are removed by subtracting. Here, subtracting gives $5x = 10$, so $x = 2$.

3. Substitute back Put $x = 2$ into $2x + 3y = 13$ to get $3y = 9$, so $y = 3$.

4. Check both equations $2(2) + 3(3) = 13$ and $3(2) + 2(3) = 12$. Both hold, so $x = 2$, $y = 3$ is the solution.

5. **Watch for a stray statement** If the last step leaves $0 = 0$, the pair has infinitely many solutions. If it leaves something false, like $0 = 9$, the pair has none.

CHECK YOURSELF

31. Solving $2x + 3y = 13$ and $3x + 2y = 12$ by elimination, the worked example multiplies the first equation by 2 and the second by 3. Why these particular numbers?

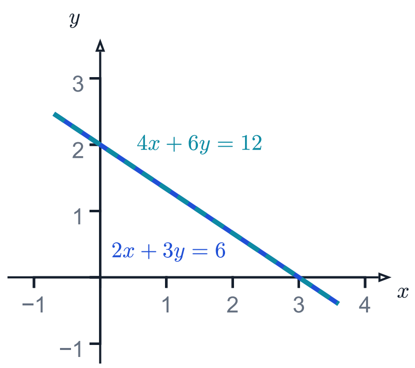
- **A.** they match the constant terms 13 and 12 as closely as possible
- **B.** they are the x-coefficients, applied to the wrong equation each time
- **C.** any two numbers would work equally well
- **D.** they make the y-coefficient equal to 6 in both equations

32. For the same pair, $2x + 3y = 13$ and $3x + 2y = 12$, which alternative multiplier pair would eliminate x instead of y?

- **A.** multiply the first equation by 3 and the second by 2
- **B.** multiply the first equation by 2 and the second by 3, as already done
- **C.** multiply both equations by 6
- **D.** multiply the first equation by 12 and the second by 13

1 more in this set online.

One line drawn twice



Multiply $2x + 3y = 6$ by 2 and you get $4x + 6y = 12$: the same line, so every point on it solves both, infinitely many. The ratios $a_1/a_2 = b_1/b_2 = c_1/c_2 = 1/2$ say the same thing.

The line 4x + 6y = 12 is drawn again, dashed, over 2x + 3y = 6, showing they are the same line.

WORKED EXAMPLE · Elimination finds infinitely many solutions

1.	$2x + 3y = 6$ and $4x + 6y = 12$	*These are the two equations we solve by elimination.*
2.	$4x + 6y = 12$	*Multiply the first equation by 2, and it now matches the second equation exactly.*
3.	$0 = 0$	*Subtracting one equation from the other leaves a statement true for every x and y.*
4.	the pair has infinitely many solutions	*An always-true statement, with no variable left, signals infinitely many solutions rather than a single answer.*

WORKED EXAMPLE · Elimination finds infinitely many solutions

5. $a_1/a_2 = b_1/b_2 = c_1/c_2 = 1/2$

CHECK ***Check the answer against the ratio test, a completely different method.*** *All three ratios match, confirming the same answer.*

RULE
If elimination reduces to $0 = 0$, stop there. That is the answer, not a sign of a mistake.

CAUTION
$0 = 0$ is not an error in your algebra. It means the two equations are the same line.

CHECK YOURSELF

33. Solving $2x + 3y = 6$ and $4x + 6y = 12$ by elimination, multiplying the first equation by 2 gives $4x + 6y = 12$, identical to the second equation. What does this signal?

- **A.** an arithmetic mistake was made somewhere in the multiplication
- **B.** the pair has no solution, since nothing new was learned
- **C.** y has been eliminated, giving a value for x alone
- **D.** the two equations describe the same line

34. Subtracting the two identical equations leaves $0 = 0$. A student concludes this means no solution. What is the error?

- **A.** $0 = 0$ is always true — infinitely many, not none
- **B.** no error; $0 = 0$ always means the pair cannot be solved
- **C.** the error is in the subtraction step itself, which should give $0 = 1$
- **D.** $0 = 0$ means exactly one solution, at $x = 0$ and $y = 0$

1 more in this set online.

WORKED EXAMPLE · A situational problem: two numbers, given their sum and difference

1. let the numbers be x and y

We name the two unknown quantities first, in words, before writing an equation.

2. $x + y = 45$ and $x - y = 5$

Translate the sum and the difference directly into one equation each.

3. $2x = 50$, so $x = 25$

Add the two equations yourself. This eliminates y directly.

4. $y = 45 - 25 = 20$

Substitute $x = 25$ back into either original equation.

5. the two numbers are 25 and 20

Name the two numbers the question asked for, in the order it asked for them.

6. $25 + 20 = 45$ and $25 - 20 = 5$

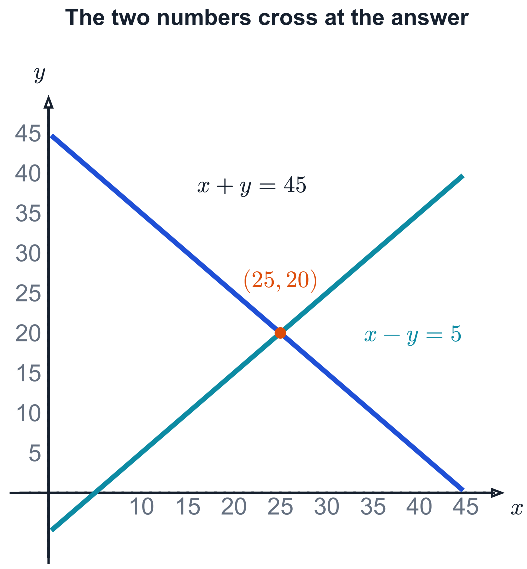
CHECK ***Check both numbers against the original conditions.*** *Both hold, so 25 and 20 really are the answer.*

BACK
Turning words into a pair of equations · p. 84

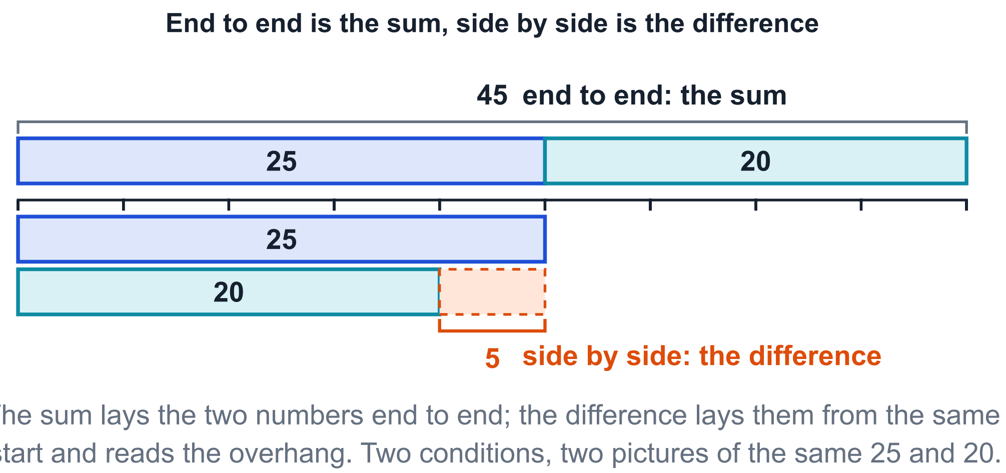
TRY
The sum of two numbers is 30 and their difference is 8. Find the numbers.

Answer: 19 and 11.

RULE
When matched coefficients are already equal and opposite, like $+y$ and $-y$ here, add the two equations directly.



The two lines from the sum and difference problem cross at 25 and 20, the numbers that satisfy both conditions.



Two bars laid end to end total 45, and laid from a shared start show a difference of 5.

TURN A WORD PROBLEM INTO A PAIR OF EQUATIONS, AND SOLVE IT.

1. **Name the unknowns** Write down, in words, what x and y stand for. Here, let x and y be the two numbers.
2. **Translate each condition** Turn each sentence into one equation. Sum 45 gives $x + y = 45$; difference 5 gives $x - y = 5$.
3. **Solve the pair** Add the two equations to eliminate y. $2x = 50$, so $x = 25$.
4. **Find the other unknown** Substitute back. $y = 45 - 25 = 20$.
5. **Check against the words** $25 + 20 = 45$ and $25 - 20 = 5$. Both conditions hold, so the two numbers are 25 and 20.

CHECK YOURSELF

35. The sum of two numbers is 45 and their difference is 5. The worked example lets the numbers be x and y and writes $x + y = 45$ and $x - y = 5$. Why two separate equations?

- **A.** one equation for x and a different, unrelated one for y
- **B.** because two numbers always need exactly two equations, regardless of the problem
- **C.** to make the arithmetic simpler, with no real relationship in the problem
- **D.** each stated condition, the sum and the difference, becomes its own equation

36. The worked example adds $x + y = 45$ and $x - y = 5$ to get $2x = 50$. What would subtracting the two equations instead give directly?

- **A.** $2y = 40$, leading to $y = 20$
- **B.** $2x = 40$, leading to $x = 20$
- **C.** $2y = 50$, leading to $y = 25$
- **D.** $0 = 40$, since x and y both cancel

1 more in this set online.

Wrapping up

IN EVERY CHAPTER

Recap

- RATIOS: $a_1/a_2 \neq b_1/b_2$ means one solution. For $x + y = 5$ and $x - y = 1$, $a_1/a_2 = 1$ and $b_1/b_2 = -1$.
- GRAPH: plot two points on each line, and read where they meet. The same pair crosses at $(3, 2)$.
- SUBSTITUTION: write one variable in terms of the other, then substitute. Solving $x + y = 10$ and $2x - y = 5$ this way gives $x = 5$, $y = 5$.
- ELIMINATION: match one variable's coefficients, then add or subtract. Solving $2x + 3y = 13$ and $3x + 2y = 12$ this way gives $x = 2$, $y = 3$.
- NO SOLUTION vs INFINITE: the third ratio decides. $6x + 3y - 15 = 0$ with $2x + y - 6 = 0$ gives no solution, $c_1/c_2 = 5/2$; $4x + 6y - 10 = 0$ with $2x + 3y - 5 = 0$ gives infinitely many, all three ratios equal to 2.

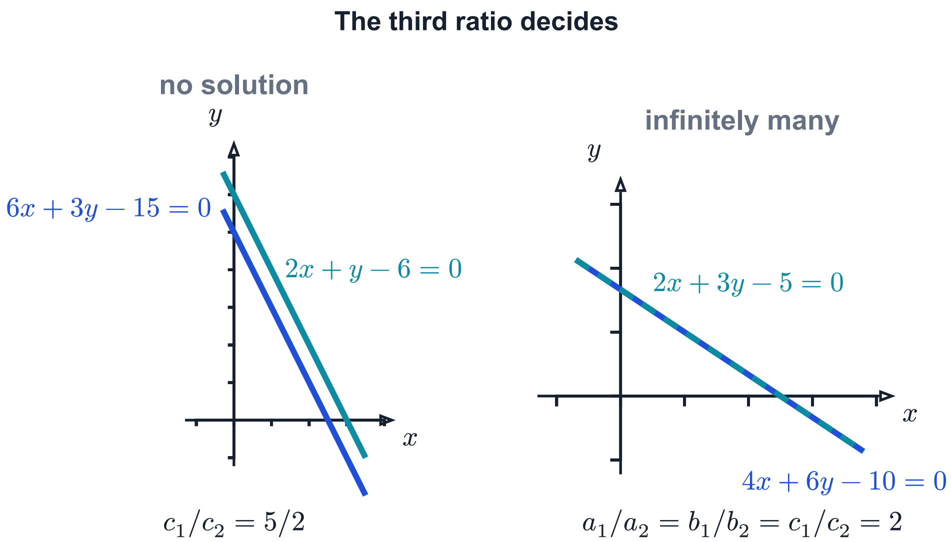
Every pair of linear equations in two variables reduces to one question: do the two lines meet at one point, never meet, or lie on top of each other?

We have now seen the same point reached three separate ways. Look back at the five rules above. The pair $x + y = 5$, $x - y = 1$ runs through the first two of them: once as a ratio comparison, once as a picture, both landing on the same point, $(3, 2)$. Every method here finds that same point, no matter which route you take to it.

Try it yourself. Pick any of the four worked pairs above, and solve it by a different method from the one used there. (Answer: you should land on the same x and y either way.)

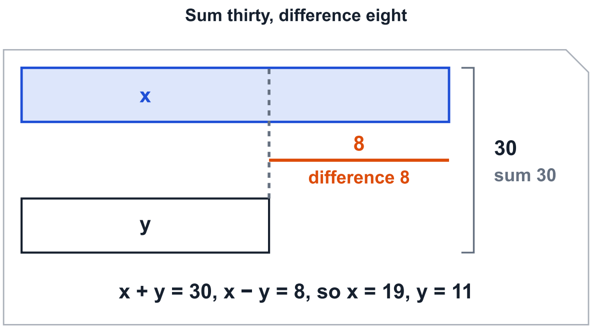
KEY
Three outcomes, three routes to find out which one: the ratios, the graph, or the algebra.

BACK
The graphical method: plot, then read · p. 77



Same first two ratios in both pairs. On the left $c_1/c_2 = 5/2$ differs, so the lines are parallel: no solution. On the right all three ratios are 2, so the lines coincide: infinitely many.

One pair of lines runs parallel with no solution, and another pair lies on the same line with infinitely many.



Two bars, x and y: laid end to end they make 30, and x overhangs y by 8. Add the two equations and the y-terms cancel, leaving x = 19, so y = 11.

Two bars, x and y, total 30 and differ by 8, giving x equals 19 and y equals 11.

CHECK YOURSELF

37. The ratio test, the graphical method, and substitution or elimination all answer the same question for a pair of linear equations. What is that question?

- **A.** do the lines meet once, never, or coincide
- **B.** which equation was written first
- **C.** how many terms each equation contains
- **D.** which variable, x or y, is more important

38. For $x + y = 9$ and $x - y = 3$, which route answers 'what are x and y' with the least work: ratio test, graphing, or elimination?

- **A.** the ratio test, since it needs no arithmetic with x and y at all
- **B.** graphing, since the crossing point can be read off directly
- **C.** elimination, since adding the equations immediately gives x
- **D.** all three take exactly the same amount of work for this pair

1 more in this set online.

CHECK YOURSELF: THE WHOLE CHAPTER

39. A pair of linear equations in x and y is really a pair of straight lines. Before drawing or solving anything, what decides whether the pair has one solution, no solution, or infinitely many?

- **A.** how large the constant terms c_1 and c_2 are on their own
- **B.** whether x or y has a larger coefficient
- **C.** comparing the ratios a_1/a_2, b_1/b_2, c_1/c_2
- **D.** how many terms each equation has

40. The substitution method solves a pair of linear equations by writing one variable in terms of the other from one equation, substituting it into the OTHER equation, then solving. If the last step gives $0 = 0$, with no variable left, what does this mean?

- **A.** There is no solution to the pair.
- **B.** A mistake was made somewhere in the algebra.
- **C.** The value of x must be 0.
- **D.** The pair has infinitely many solutions.

3 more in this set online.

IN EVERY CHAPTER

Where you will meet this

Two cost lines for Class A and Class B cross at 3 months and ₹1400, after which Class B stays cheaper.

You will compare two costs for the rest of your life: two plans, two shops, two prices. Here are seven of those places.

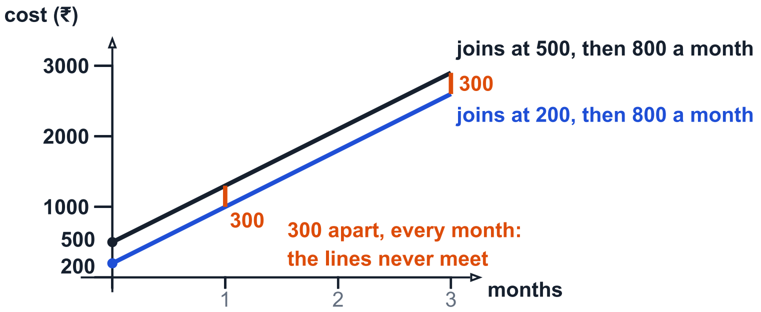
- **Watching a savings plan grow.** You start with ₹500 saved. You add ₹200 every week after that. After x weeks you have $y = 200x + 500$ rupees.
  The maths: every (x, y) pair that fits this equation lies on one straight line, out of infinitely many such pairs.
  At $x = 6$ weeks: $y = 200 \cdot 6 + 500 = 1700$ rupees.
- **Comparing two tuition fee plans.** Class A charges ₹500 joining plus ₹300 a month. Class B charges ₹800 joining plus ₹200 a month.
  The maths: since the two monthly rates differ, the ratio test says there is exactly one month

when both cost the same.
At $x = 3$ months: Class A costs $300 \cdot 3 + 500 = 1400$ and Class B costs $200 \cdot 3 + 800 = 1400$.

- **Comparing two gym memberships.** Two gyms both charge ₹800 a month. Their joining fees are ₹200 and ₹500.
  The maths: the two lines are parallel, same rate and different start, so the pair has no solution.
  For any x months: $(800x + 500) - (800x + 200) = 300$ rupees, every time.

BACK
One equation, one line, endless solutions · p. 72

Two gyms stay three hundred apart



Same monthly rate means the same slope, so the two cost lines are parallel; the one that joins ₹300 dearer stays ₹300 dearer every month, and there is no month when the gyms cost the same.

A shared rate is what makes a pair unsolvable: with the slopes equal the joining-fee gap never closes, so no month answers the question.

- **Two bills that tell you nothing new.** One bill says 2 pens and 3 notebooks cost ₹120. A friend's bill says 4 pens and 6 notebooks cost ₹240.
  The maths: all three ratios are equal, so the pair has infinitely many solutions. You cannot find the price of one pen.
  Here $2/4 = 3/6 = 120/240$. Pens at ₹30 and notebooks at ₹20 fit both bills. So do pens at ₹15 and notebooks at ₹30.

- **Counting cash in a till.** Your cash box has 25 notes of ₹10 and ₹20 only, worth ₹380 in all.
  The maths: write one note count in terms of the other, then substitute, to find how many of each.
  Let x be ₹10 notes: $10x + 20(25 - x) = 380$ gives $x = 12$. So 12 notes of ₹10 and 13 of ₹20.

- **Blending rice to a target price.** A grocer mixes rice at ₹40 a kg and ₹60 a kg to make 10 kg costing ₹520 in all.
  The maths: match the coefficients of one variable, then subtract, to find how many kg of each grade.
  Multiply $x + y = 10$ by 40: $40x + 40y = 400$. Subtract from $40x + 60y = 520$: $20y = 120$, so $y = 6$ kg.

- **Printing posters for a fest.** A print shop prints 40 posters for a college fest. A2 size costs ₹120 each and A3 costs ₹60 each, for ₹3600 in all.
  The maths: name the two unknown counts as x and y, turn each fact into one equation, then solve.
  Let x be the count of A2 posters: $120x + 60(40 - x) = 3600$ gives $x = 20$. So 20 of each size.

Your turn. A shop has 25 shirts, some priced ₹150 and some ₹300, worth ₹6000 in all. How many of each price? *Answer:* Let x be ₹150 shirts: $150x + 300(25 - x) = 6000$ gives $x = 10$. So 10 shirts at ₹150 and 15 at ₹300.

Practice: Exercise 3.1

PRACTICE SET

1. PRACTICE Solve $x + y = 7$ and $x - y = 3$ graphically. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1.	$x + y = 7$ passes through $(0, 7)$ and $(7, 0)$	*put $x = 0$ to get one point, then $y = 0$ to get the other*
2.	$x - y = 3$ passes through $(0, -3)$ and $(3, 0)$	*the same two substitutions on the second equation*
3.	the two lines cross at $(5, 2)$	*joining each pair of points draws the lines, and they meet at one point*
4.	$5 + 2 = 7$ and $5 - 2 = 3$	*the check — the crossing point must satisfy both equations*

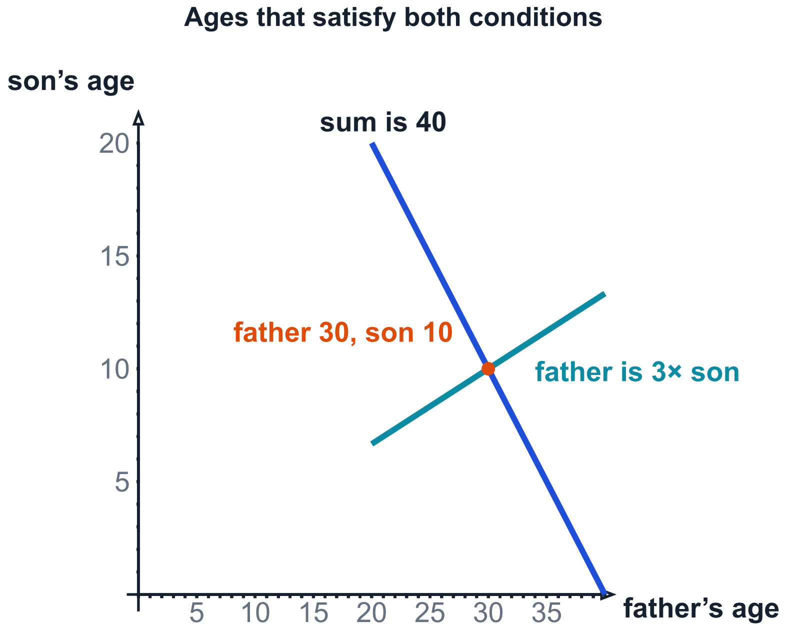
2. PRACTICE On comparing the ratios a_1/a_2, b_1/b_2 and c_1/c_2, state whether the lines represented by $2x - y - 5 = 0$ and $4x - 2y - 6 = 0$ are intersecting, parallel, or coincident. (Start the same way — write all three ratios down before deciding anything.)
3. PRACTICE On comparing the same three ratios, state whether the lines represented by $3x - 2y - 7 = 0$ and $x + y - 4 = 0$ are intersecting, parallel, or coincident. (Two ratios are enough to settle this one.)
4. PRACTICE The sum of a father's age and his son's age is 40 years, and the father's age is three times the son's age. Represent this situation as a pair of linear equations and solve it graphically.
5. PRACTICE On comparing the ratios a_1/a_2, b_1/b_2 and c_1/c_2, state whether the lines represented by $2x + 3y - 9 = 0$ and $4x + 6y - 18 = 0$ are intersecting, parallel, or coincident.
6. PRACTICE On comparing the same three ratios, state whether the lines represented by $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$ are intersecting, parallel, or coincident.
7. PRACTICE On comparing the same three ratios, state whether the lines represented by $3x + y - 8 = 0$ and $x - y = 0$ are intersecting, parallel, or coincident.

ANSWERS

1. The lines cross at $(5, 2)$, so $x = 5$ and $y = 2$. Substituting back gives $5 + 2 = 7$ and $5 - 2 = 3$, so both equations hold.
2. $a_1/a_2 = b_1/b_2 = 1/2$ but $c_1/c_2 = 5/6$, so the lines are parallel and the pair has no solution.

3. $a_1/a_2 = 3$ and $b_1/b_2 = -2$, and $3 \neq -2$, so the lines intersect and the pair has a unique solution.
4. Son's age y, father's age x: $x + y = 40$, $x = 3y$. Solved graphically, the lines cross at $(30, 10)$ — the father is 30, the son is 10.
5. $a_1/a_2 = b_1/b_2 = c_1/c_2 = 1/2$: coincident, infinitely many solutions.
6. $a_1/a_2 = b_1/b_2 = 1/2$, but $c_1/c_2 = 1/3$: parallel, no solution.
7. $a_1/a_2 = 3$, $b_1/b_2 = -1$: intersecting, a unique solution.
Full solutions: **manthika.com/learn/math10-ch03**

Further practice



The father's age line and the son's age line cross at a father of 30 and a son of 10.

PRACTICE SET

8. PRACTICE On comparing the ratios a_1/a_2, b_1/b_2 and c_1/c_2, state whether the lines represented by $x + y - 7 = 0$ and $2x - y - 2 = 0$ are intersecting, parallel, or coincident.

9. PRACTICE On comparing the same three ratios, state whether the lines represented by $4x - 5y + 7 = 0$ and $8x - 10y + 3 = 0$ are consistent or inconsistent.

12. PRACTICE Half the perimeter of a rectangular field is 28 m, and the length is 6 m more than the breadth. Form a pair of linear equations for this situation and solve it graphically to find the field's length and breadth.

13. PRACTICE For what value of k do the equations $2x + ky = 5$ and $4x + 6y = 9$ represent a pair of parallel lines?

14. PRACTICE Write one linear equation that, together with $3x + 2y - 12 = 0$, forms a pair of coincident lines.

10. PRACTICE Which of these best describes the pair $3x - y - 5 = 0$ and $6x - 2y - 7 = 0$?

A. Intersecting lines, a unique solution
B. Parallel lines, no solution
C. Coincident lines, infinitely many solutions
D. Cannot be decided without solving

11. PRACTICE A shop sells 3 pens and 2 pencils together for ₹47, and 4 pens and 3 pencils together for ₹64. Form a pair of linear equations for this situation and solve it graphically to find the cost of one pen and one pencil.

15. PRACTICE The cost of 2 tables and 3 chairs is ₹4100, and the cost of 3 tables and 2 chairs is ₹4400. Using the ratio test, check first whether this pair of equations has a unique solution, then solve it graphically to find the cost of one table and one chair.

ANSWERS

8. $a_1/a_2 = 1/2$, $b_1/b_2 = -1$: since $1/2 \neq -1$, the lines are intersecting and the pair has a unique solution.
9. $a_1/a_2 = b_1/b_2 = 1/2$, but $c_1/c_2 = 7/3$: the lines are parallel, so the pair is inconsistent.
10. B — Parallel lines, no solution.
11. $3x + 2y = 47$, $4x + 3y = 64$: the lines cross at $(13, 4)$ — one pen costs ₹13, one pencil costs ₹4.
12. $x + y = 28$, $x - y = 6$: the lines cross at $(17, 11)$ — the field's length is 17 m and its breadth is 11 m.
13. $k = 3$.
14. Any non-zero multiple works, since the ratios a_1/a_2, b_1/b_2, c_1/c_2 must all be equal — for example, $6x + 4y - 24 = 0$, obtained by multiplying every term by 2.
15. $a_1/a_2 = 2/3$ and $b_1/b_2 = 3/2$ are unequal, so the pair has a unique solution: the lines cross at $(1000, 700)$ — one table costs ₹1000, one chair costs ₹700.
Full solutions: **manthika.com/learn/math10-ch03**

Practice: Exercise 3.2

PRACTICE SET

1. PRACTICE Solve $x + y = 12$ and $3x - y = 8$ by the substitution method. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1.	$x = 12 - y$	*rearrange the first equation to get one variable on its own*
2.	$3(12 - y) - y = 8$	*substitute that expression into the second equation, so only y is left*
3.	$36 - 4y = 8$, so $y = 7$	*expand and collect the y terms, then solve the single equation*

SOLUTION — Q1 · **Worked in full**

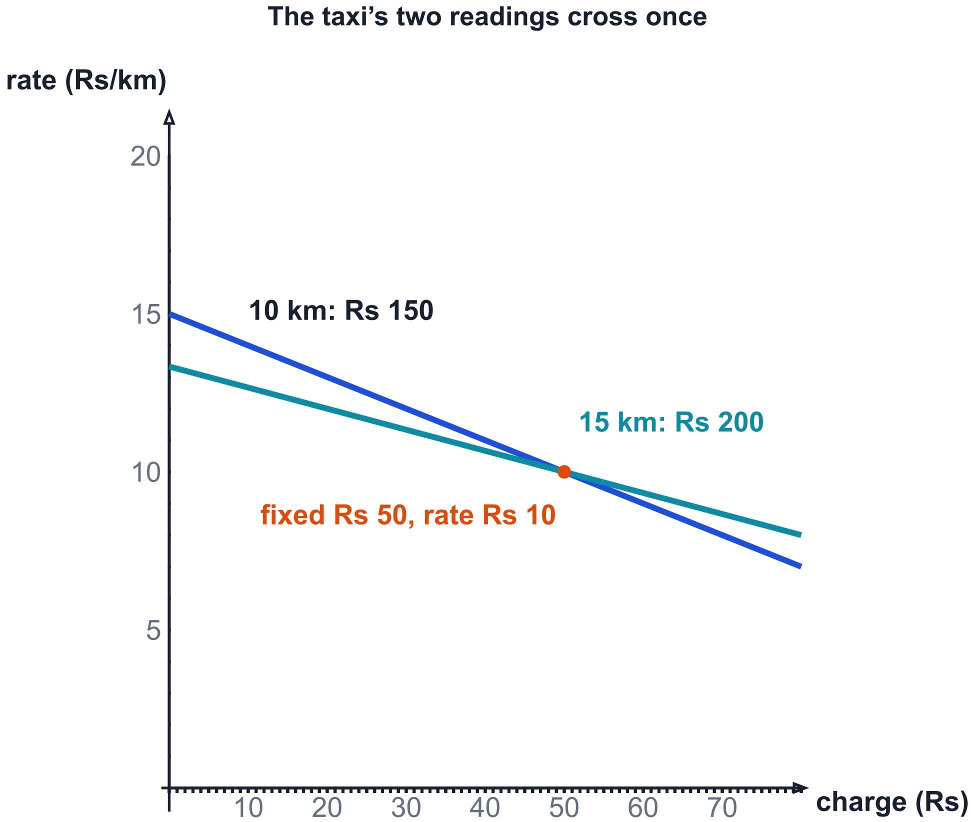
4.	$x = 12 - 7 = 5$	*substitute back — the answer to a pair is a pair, not one number*
5.	$5 + 7 = 12$ and $3(5) - 7 = 8$	*the check — both original equations must hold*

2. **PRACTICE** The sum of two numbers is 30, and one of them is four times the other. Find the two numbers. (Start the same way — let the numbers be x and y, write the pair $x + y = 30$ and $x = 4y$, then substitute.)
3. **PRACTICE** A pen and a notebook together cost ₹90, and the notebook costs ₹30 more than the pen. Find the cost of each. (Name the two costs first, then substitute.)
4. **PRACTICE** The difference between two numbers is 26, and one number is three times the other. Find the two numbers.
5. **PRACTICE** Two angles are supplementary. The larger angle exceeds the smaller by 18°. Find both angles.
6. **PRACTICE** 3 bats and 6 balls together cost ₹1800, while 5 bats and 3 balls together cost ₹2650. Find the cost of one bat and one ball.
7. **PRACTICE** A taxi charges a fixed amount plus a rate per kilometre travelled. A 10-kilometre ride costs ₹150, and a 15-kilometre ride costs ₹200. Find the fixed charge and the rate per kilometre.
8. **PRACTICE** A fraction becomes 1 when 1 is added to its numerator, and becomes $1/2$ when 1 is added to its denominator instead. Find the fraction.
9. **PRACTICE** Five years ago, a mother was five times as old as her daughter. Ten years from now, she will be twice as old as her daughter will be then. Find their present ages.

ANSWERS **1.** $x = 5$ and $y = 7$. **2.** The numbers are 24 and 6.
3. The pen costs ₹30 and the notebook ₹60.
4. $x - y = 26$, $x = 3y$: the numbers are 39 and 13.
5. $x + y = 180$, $x - y = 18$: the angles are 99° and 81°.
6. $x + 2y = 600$, $5x + 3y = 2650$: one bat costs ₹500, one ball costs ₹50.
7. $x + 10y = 150$, $x + 15y = 200$: fixed charge ₹50, rate ₹10 per kilometre.
8. $x + 1 = y$, $2x = y + 1$: the fraction is $2/3$.
9. $x - 5 = 5(y - 5)$, $x + 10 = 2(y + 10)$: mother 30, daughter 10.
Full solutions: **manthika.com/learn/math10-ch03**

BACK
The substitution method, step by step · p. 81

Further practice



The two ride conditions become lines that cross at a fixed charge of 50 rupees and a rate of 10 rupees per kilometre.

PRACTICE SET

10. PRACTICE By the substitution method, what is the solution of $x + y = 6$ and $x - y = 2$?

A. $(3, 3)$
B. $(4, 2)$
C. $(2, 4)$
D. $(5, 1)$

SOLUTION — Q10 · **Worked in full**

1.	$x = y + 2$	*write x in terms of y, from the second equation*
2.	$(y + 2) + y = 6$	*substitute into the first equation*
3.	$2y = 4$	*simplify*
4.	$y = 4/2 = 2$	*divide to find y*
5.	$x = 2 + 2 = 4$	*substitute back to find x*

11. PRACTICE One number is 4 more than twice another number, and their sum is 25. Find the two numbers.

12. PRACTICE Two angles are complementary. The larger angle exceeds the smaller by 24°. Find both angles.

15. PRACTICE A mobile plan charges a fixed monthly rent plus a fixed rate per minute of calls. A month with 100 minutes of calls costs ₹250, and a month with 160 minutes costs ₹310. Find the fixed rent and the rate per minute.

13. PRACTICE Solve the pair $3x - 2y = 11$ and $4x + 3y = 26$ using the substitution method.

14. PRACTICE The cost of 4 kg of apples and 3 kg of oranges is ₹440, while the cost of 2 kg of apples and 5 kg of oranges is ₹360. Find the cost per kilogram of each fruit.

16. PRACTICE In a test of 20 questions, each correct answer earns 3 marks and each wrong answer loses 1 mark. Rhea attempted every question and scored a total of 40 marks. Find the number of questions she answered correctly and the number she got wrong.

17. PRACTICE Two numbers are in the ratio 3 : 5. If 5 is added to each number, the ratio becomes 2 : 3. Find the numbers.

ANSWERS **10.** B — $(4, 2)$. **11.** $x = 2y + 4$, $x + y = 25$: the numbers are 18 and 7.
12. $x + y = 90$, $x - y = 24$: the angles are 57° and 33°. **13.** $x = 5$, $y = 2$.
14. $4x + 3y = 440$, $2x + 5y = 360$: apples cost ₹80 per kg, oranges cost ₹40 per kg.
15. $x + 100y = 250$, $x + 160y = 310$: the fixed rent is ₹150, and the rate is ₹1 per minute.
16. $x + y = 20$, $3x - y = 40$: she answered 15 questions correctly and 5 wrongly.
17. $y = 5x/3$, $3(x + 5) = 2(y + 5)$: the numbers are 15 and 25.
Full solutions: **manthika.com/learn/math10-ch03**

Practice: Exercise 3.3

PRACTICE SET

1. PRACTICE Solve $3x + 2y = 16$ and $2x + 3y = 14$ by elimination. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1.	$9x + 6y = 48$ and $4x + 6y = 28$	*multiply the first equation by 3 and the second by 2, so the y terms match in size*
2.	both matched terms are $+6y$, so subtract	*same sign cancels under subtraction; opposite signs would need addition instead*
3.	$5x = 20$, so $x = 4$	*subtracting removes y and leaves one equation in x*
4.	$3(4) + 2y = 16$, so $y = 2$	*substitute back into either original equation*
5.	$2(4) + 3(2) = 14$	*the check — try the pair in the equation that was not used to find y*

2. PRACTICE Solve $4x + y = 14$ and $2x - y = 4$ by elimination. (Start the same way — the y terms already match in size, at $+1$ and -1. Read those signs before you choose the operation.)

6. PRACTICE The present age of a father is six years more than three times his son's age. Three years from now, the father's age will be ten years more than twice the son's age. Find their present ages.

3. PRACTICE The sum of the digits of a two-digit number is 9, and the number is 27 more than the number formed by reversing its digits. Find the number. (Call the tens digit x and the units digit y, so the number is $10x + y$.)

4. PRACTICE Solve the pair $2x + 3y = 11$ and $2x - 4y = -24$, once by elimination and once by substitution.

5. PRACTICE A fraction becomes $9/11$ when 2 is added to both its numerator and denominator, and becomes $5/6$ when 3 is added to both instead. Find the fraction.

7. PRACTICE A two-digit number is 4 more than 6 times the sum of its digits. Subtracting 18 from the number reverses its digits. Find the number.

8. PRACTICE A person has only ₹50 and ₹100 notes. She has 25 notes in all, worth ₹1750 together. Find how many notes of each kind she has.

9. PRACTICE A lending library charges a fixed amount for the first three days a book is kept, and a further amount for each day after that. Keeping a book for 4 days costs ₹27; keeping it for 7 days costs ₹45. Find the fixed charge and the daily charge beyond the first three days.

ANSWERS **1.** $x = 4$ and $y = 2$.
2. $x = 3$ and $y = 2$ — the matched terms have opposite signs, so the equations are added, not subtracted.
3. The number is 63. **4.** $x = -2$, $y = 5$, by both methods.
5. $11x - 9y = -4$, $6x - 5y = -3$: the fraction is $7/9$.
6. $x = 3y + 6$, $x = 2y + 13$: father 27, son 7.
7. $4x - 5y = 4$, $x - y = 2$: the number is 64.
8. $x + y = 25$, $x + 2y = 35$: 15 notes of ₹50 and 10 notes of ₹100.
9. $x + y = 27$, $x + 4y = 45$: fixed charge ₹21, daily charge ₹6.
Full solutions: **manthika.com/learn/math10-ch03**

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Further practice

PRACTICE SET

10. PRACTICE Solve the pair $3x + 2y = 12$ and $5x - 2y = 4$, once by elimination and once by substitution.

SOLUTION — Q10 · **Worked in full**

1.	$8x = 16$	*add the two equations directly — this eliminates y since its coefficients are already opposite, $+2y$ and $-2y$*
2.	$x = 16/8 = 2$	*divide to find x*
3.	$2y = 12 - 3(2) = 6$	*substitute $x = 2$ back into the first equation*
4.	$y = 6/2 = 3$	*divide to find y*

11. PRACTICE A person has ₹20 and ₹10 notes only, 30 notes in all, worth ₹450 together. How many ₹20 notes does she have?

A. 10
B. 15
C. 20
D. 25

12. PRACTICE The sum of the digits of a two-digit number is 12. Adding 18 to the number reverses its digits. Find the number.

13. PRACTICE Five years ago, an uncle was three times as old as his nephew. Ten years from now, he will be twice as old as his nephew will be then. Find their present ages.

14. PRACTICE A boat travels 30 km downstream in 3 hours, and the same 30 km upstream in 5 hours. Form a pair of linear equations in the boat's speed in still water and the speed of the stream, and solve them.

15. PRACTICE A fraction reduces to $2/3$ when 2 is subtracted from both its numerator and denominator, and reduces to $3/4$ when 1 is added to both instead. Find the fraction.

ANSWERS

10. $x = 2$, $y = 3$, by both elimination (adding the equations directly) and substitution (writing x in terms of y from either equation).

11. B — 15. **12.** $x + y = 12$, $y - x = 2$: the number is 57.

13. $x - 3y = -10$, $x - 2y = 10$: the uncle is 50 and the nephew is 20.

14. $x + y = 10$, $x - y = 6$: the boat's speed in still water is 8 km/h and the stream's speed is 2 km/h.

15. $3x - 2y = 2$, $4x - 3y = -1$: the fraction is $8/11$.

Full solutions: **manthika.com/learn/math10-ch03**