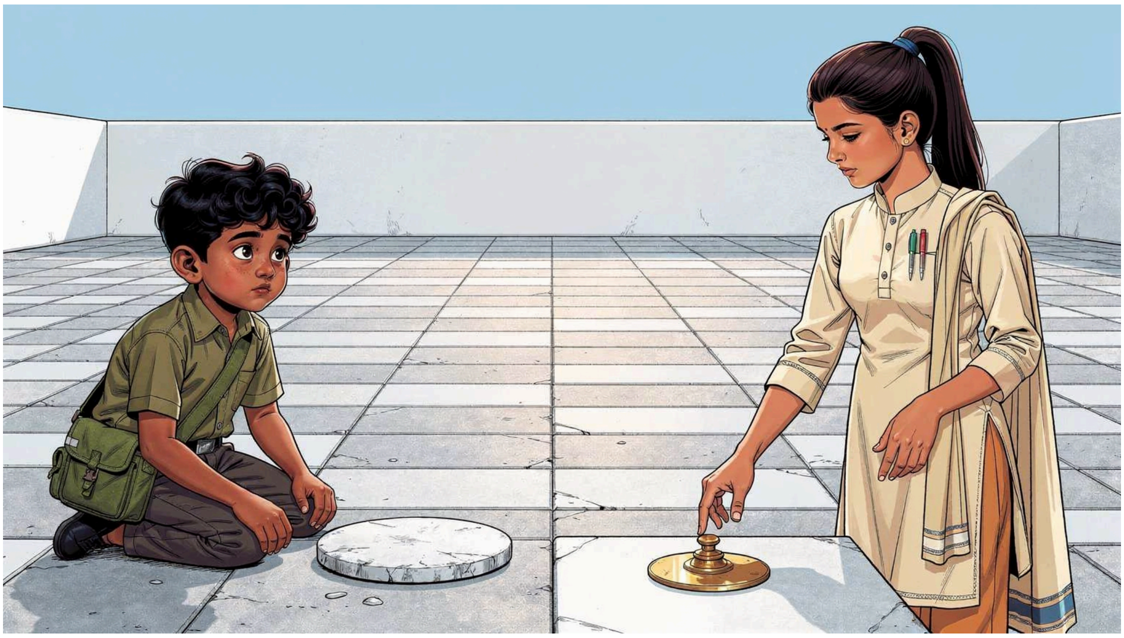
7 CHAPTER 7 Coordinate Geometry

WHAT THIS CHAPTER BUILDS

You will learn to find the distance between two points, and the point that divides them in a given ratio, using only their coordinates.

COORDINATE GEOMETRY UNIT

IN THIS CHAPTER



Kabir and Ananya set a pale disc and a brass disc on a tiled floor, counting the tiles to each from one corner.

Getting started

Geometry with coordinates

A STORY

Two discs on the tiles

Kabir kneels on the tiled terrace beside a pale disc. Ananya stands a few tiles away and sets a brass disc down on the floor.

"Count the tiles to each disc, from that corner," Ananya says.

Kabir counts to the pale disc. "3 across, 2 up." He counts to the brass disc. "7 across, 6 up."

"Two numbers each, and you know exactly where both discs sit," Ananya says.

"But how far apart are they?" Kabir asks. "I do not want to lay a tape across the whole floor."

"You do not need to," Ananya says. "The tile counts alone give you the distance, without ever touching a tape to the floor."

"Just from four numbers?" Kabir asks.

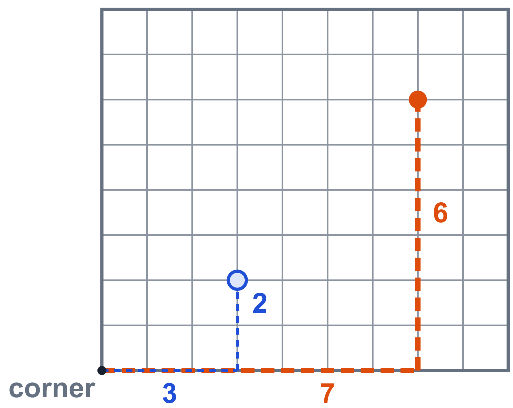
"Just from four numbers," Ananya says.

You will learn the rule that turns those four numbers into one distance. It works for any two points on a floor like this.

RULE
Label the two points (x_1, y_1) and (x_2, y_2) before touching a formula. Swap a label, and the answer swaps too.

KEY
A ruler measures a plotted length. Coordinates alone can find that same length, with nothing drawn.

Two discs on the tiles



Counted from the marked corner, the pale disc is three across and two up, and the brass disc is seven across and six up.

Four counts fix two points, and the distance formula turns those four counts into one length.

Every point on a flat map can be pinned down by two numbers. Once it has those two numbers, we can find the distance between two points without a ruler.

Picture a district grid. Streets run east to west and north to south. Each street is numbered out from a fixed corner. Two towns sit at two of those grid points, joined by one straight road. Walk from one town to the other and you cross 3 streets going east and 4 going north, and the road itself is 5 long. One rule does that for any two towns, and the next section builds it.

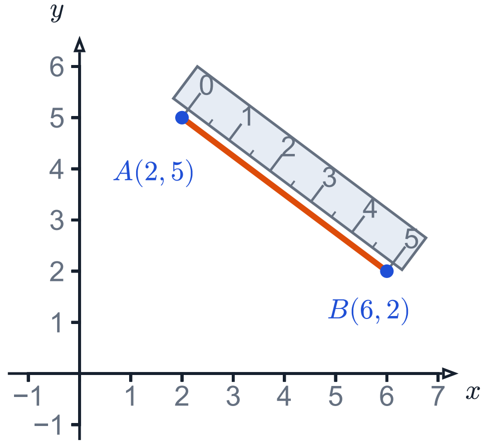
A point can also sit ON that road, instead of being measured across it. Cut the road so that one piece is twice as long as the other, and ask where the cut falls. The same handful of numbers answers that too, through a second rule, a few pages further on.

Two jobs, one coordinate picture. Every section from here on puts one of them to work.

Your turn: two towns sit at $(2, 5)$ and $(9, 5)$. How far apart are they? (Answer: 7 units, since they share a y-coordinate, and the gap is $9 - 2$.)

The ruler and the formula agree

plotted, then measured



AB = 5 units

the two pairs, nothing drawn

$A(2, 5)$ $B(6, 2)$

$$\begin{aligned} AB &= \sqrt{(6-2)^2 + (2-5)^2} \\ &= \sqrt{4^2 + (-3)^2} \\ &= \sqrt{16+9} = \sqrt{25} \end{aligned}$$

AB = 5 units

A ruler laid along the plotted segment reads 5 units, and the two coordinate pairs give the same 5 with nothing drawn at all.

A SCHOLAR INDIA REMEMBERS

Brahmagupta worked in a school of astronomy with a careful habit. A calculation said where something would be in the sky. The astronomers checked it against the sky. When the two did not match, they corrected the numbers in the calculation. They did not argue with what they saw. You can do the same here. Work out a distance with the formula, then look at the graph. If the two disagree, fix the working.

Brahmagupta · the brahminy kite

IN THE WILD

Two pins dropped on a phone's map, one at home and one at school, give the app enough to state the straight-line distance between them, straight from their coordinates. No ruler on the screen. No line drawn. The same idea works on paper: the distance between any two plotted points, or the point a set fraction of the way between them, comes from their four numbers alone.

CHECK YOURSELF

1. With points written as coordinate pairs, this chapter's opening claim hands you two results directly, once the points are known. What are they?

- **A.** the slope of the line through two points
- **B.** a length, and a dividing point
- **C.** the area enclosed by three or four points
- **D.** the equation of the line joining two points

2. Both formulas in this chapter's frame — the length between two points and the dividing point for a ratio — are built from the same two ingredients. What are those ingredients?

- **A.** just the two points' x- and y-coordinates
- **B.** the two points' coordinates plus the slope of the segment joining them
- **C.** the two points' coordinates plus the angle the segment makes with the x-axis
- **D.** a diagram drawn to scale, read off with a ruler

1 more in this set online.

IN EVERY CHAPTER

Before you start

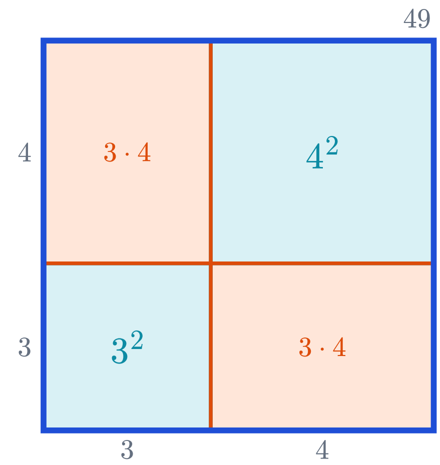
DO THIS FIRST

You have plotted points before. Now measure between them. Try each check below. It takes a minute.

- **A point's two distances** (Class 9, Ganita Manjari, page 14).
  Here: every point's x and y are read off as these two distances.
  Check: Point $(-4, 7)$ sits 4 units from the y-axis and 7 from the x-axis.
- **The Pythagoras theorem** (Class 8, Ganita Prakash Part 2, page 87).
  Here: the distance formula applies this same theorem on the grid.
  Check: Legs 24 and 32 give hypotenuse $\sqrt{24^2 + 32^2} = 40$.
- **Dividing a whole in a ratio** (Class 8, Ganita Prakash Part 2, page 58).
  Here: the section formula divides a segment in a given ratio, just like this.
  Check: 12 divided in ratio $2 : 1$ gives 8 and 4, since $8 : 4$ is $2 : 1$.
- **A ratio in simplest form** (Class 8, Ganita Prakash Part 1, page 162).
  Here: a found ratio here is confirmed by reducing it to simplest form.
  Check: $4 : 2$ and $10 : 5$ both reduce to $2 : 1$, so they are in proportion.
- **Equal side lengths** (Class 8, Ganita Prakash Part 1, page 99).
  Here: a figure here is named from its equal sides, just like this one.
  Check: A quadrilateral with all four sides 5 has perimeter $4 \cdot 5 = 20$.
- **Squaring a sum** (Class 9, Ganita Manjari, page 70).
  Here: each distance, squared to drop its root, expands like this.
  Check: $(3 + 4)^2 = 3^2 + 2 \cdot 3 \cdot 4 + 4^2 = 49$.

If any of these felt new, read the page named before going on.

Seven squared in four pieces



The square on 3 + 4 is the square on 3, the square on 4,
and two rectangles each 3 by 4: 49 without multiplying.

A square on a sum splits into two squares and two rectangles, and that expansion is what every squared distance in this chapter unpacks into.

Locating a point by coordinates

A point in a plane is pinned down by an ordered pair of coordinates, (x, y).

The x-coordinate is the point's distance from the y-axis. The y-coordinate is its distance from the x-axis. Get this pairing backwards, and every formula built on it points at the wrong spot.

Try it yourself before reading on. What are the coordinates of a point that sits 4 units from the y-axis, and none from the x-axis? Such a point sits on the x-axis itself, at $(4, 0)$. Now swap the reasoning. $(0, 4)$ is a different point, sitting on the y-axis instead.

The naming looks backwards until we have used it a few times. The axis a coordinate is measured FROM is not the axis that shares its name.

Your turn: a point sits 3 units from the x-axis and 7 units from the y-axis, in the first quadrant. What are its coordinates? (Answer: $(7, 3)$, since x is measured from the y-axis and y from the x-axis.)

TRY
Coordinates of a point on the x-axis, 4 units from the origin?

Answer: $(4, 0)$.

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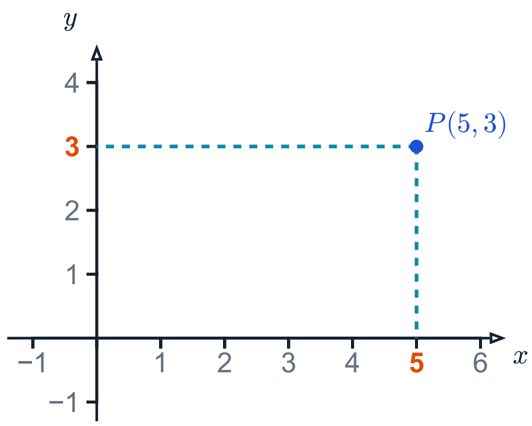
KEY
The x-coordinate is the distance from the y-axis; the y-coordinate is the distance from the x-axis.

Locating a point by its coordinates



The point at 5 and 3 has dashed lines showing 5 as its distance from the y-axis, and 3 from the x-axis.

A SCHOLAR INDIA REMEMBERS

Where is $(-3,2)$? Is -3 a real place to stand? Yes, it is. Start at the origin. Move 3 units to the left, because the number is negative. Then move 2 units up. A negative coordinate is still a place. It is on the other side of the origin.

Brahmagupta · the brahminy kite

CHECK YOURSELF

3. For a point (x, y), the x-coordinate is the point's distance from

- **A.** the y-axis
- **B.** the x-axis
- **C.** the origin

4. A student says: "for the point $(3,5)$, the 3 tells me how high up it is, and the 5 tells me how far right." What is wrong with this?

- **A.** nothing is wrong — the student has it right
- **B.** both numbers actually measure the same distance, from the origin
- **C.** the two are swapped
- **D.** the point should be written as $(5,3)$ instead

1 more in this set online.

Dividing a segment in a given ratio

A point P divides the segment joining A and B internally when P sits between the two points, on the straight road connecting them.

The ratio $m:n$ compares P's distance from A to its distance from B, in that order. Picture a ratio of $1:1$. It places P exactly halfway. Now picture $1:4$ instead. P sits close to A, since its share of the distance is the smaller number.

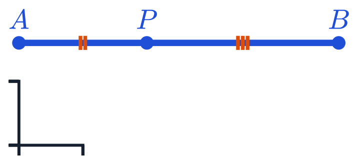
Check this on any ratio. P must sit between A and B for the division to be internal. A point outside the segment divides it externally instead.

Your turn: P divides the segment joining A and B in the ratio $1:3$. Which point does P sit closer to? (Answer: A, since the ratio names P's distance from A first, and 1 is the smaller share.)

RULE
Internal means P lands inside: $2:3$ cuts the segment into 5 parts and puts P two of them along from A.

KEY
$1:3$ means AP is a third of PB, so P sits nearer A.

What dividing in a ratio means



Two tick marks on one piece of the segment, and three on the other, show the ratio 2 to 3, with no coordinates yet.

CHECK YOURSELF

5. Point P divides segment AB internally in the ratio $3:2$. This means

- **A.** P sits exactly $3/2$ of the way along AB, measured from A
- **B.** P's distance from A compares to its distance from B as 3 is to 2
- **C.** segment AP is always exactly 3 units and PB is always exactly 2 units
- **D.** P lies exactly at the midpoint of AB

6. A student claims: "in the ratio $1:3$, point P must be closer to B than to A, since 3 is the bigger number." What is wrong with this?

- **A.** nothing is wrong — P is indeed closer to B
- **B.** the ratio needs both numbers to be equal before you can compare distances at all
- **C.** P is actually closer to A
- **D.** P must lie outside the segment AB entirely

1 more in this set online.

The ideas

The distance between two points

Two towns on a district grid sit at (x_1, y_1) and (x_2, y_2). What is the straight-line distance between them?

We draw a horizontal line through one town, and a vertical line through the other. The two lines meet at a third point. The three points form a right triangle. Its two legs measure $|x_2 - x_1|$ and $|y_2 - y_1|$, one horizontal and one vertical. The road joining the two towns is the hypotenuse of that triangle.

The Pythagoras theorem gives us that hypotenuse directly. **The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, no matter where the two towns sit.**

What is the distance between $(0, 0)$ and $(3, 4)$? Substitute directly, and you get $\sqrt{3^2 + 4^2} = 5$.

Subtracting inside either bracket in the opposite order changes nothing. $(x_1 - x_2)^2$ and $(x_2 - x_1)^2$ are equal, because squaring removes a negative sign either way.

One more thing the formula settles on its own. Three points are COLLINEAR when they sit on one straight line. Take the three distances between them, two points at a time. If the two shorter ones add up to the longest, there is no bend at the middle point and the three are collinear. If they add up to more than the longest, the middle point sits off the line. Take $(0, 0)$, $(3, 4)$ and $(6, 8)$: the three distances are 5, 5 and 10, and $5 + 5 = 10$, so those three are collinear.

Your turn: find the distance between $(2, 2)$ and $(8, 10)$. (Answer: 10, since $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$.)

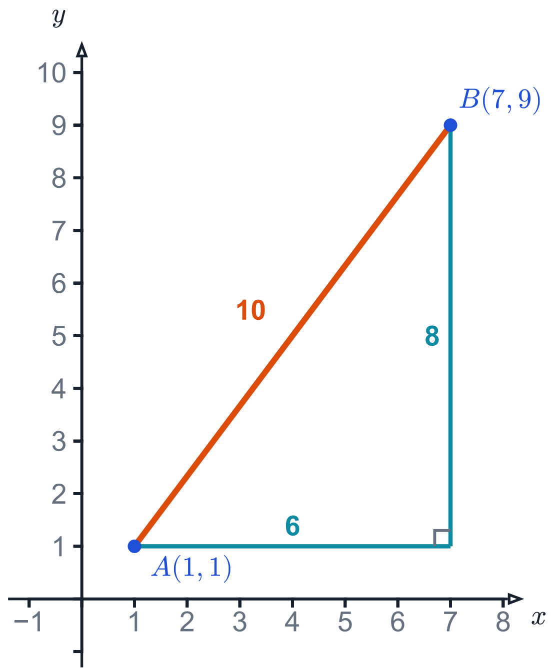
TRY
Distance between $(0, 0)$ and $(3, 4)$?

Answer: $\sqrt{3^2 + 4^2} = 5$.

RULE
Subtract in either order inside each square; squaring removes the sign either way.

KEY
One formula, from the Pythagoras theorem: legs $|x_2 - x_1|$ and $|y_2 - y_1|$.

The legs are parallel to the axes



A right triangle on the grid has legs 6 and 8 at a right angle, and a diagonal of 10 as the distance.

CHECK YOURSELF

7. The distance between (x_1, y_1) and (x_2, y_2) is

- **A.** $(x_2 - x_1)^2 + (y_2 - y_1)^2$
- **B.** $|x_2 - x_1| + |y_2 - y_1|$
- **C.** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- **D.** $\sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$

8. A student finds the distance between $(2, 3)$ and $(5, 7)$ by writing $\sqrt{3 + 4} = \sqrt{7}$. What went wrong?

- **A.** nothing — $\sqrt{7}$ is the correct distance
- **B.** the square root should not have been taken at all
- **C.** the two points were subtracted in the wrong order
- **D.** the differences were not squared first

3 more in this set online.

Distance from the origin

The origin $O(0,0)$ is a point like any other. The distance formula still applies to it directly.

Set $x_1 = y_1 = 0$ in the general formula yourself. One whole term drops out of each bracket. **The distance of a point (x, y) from the origin comes out to $\sqrt{x^2 + y^2}$.** This is the same formula, with one point fixed at the origin.

Check it on $(6, 8)$. You get $\sqrt{6^2 + 8^2} = 10$ units from the origin, found the same way as the distance between any other two points.

Your turn: find the distance of $(7, 24)$ from the origin. (Answer: 25, since $\sqrt{7^2 + 24^2} = \sqrt{625} = 25$.)

TRY
Distance of $(6, 8)$ from the origin?

Answer: $\sqrt{6^2 + 8^2} = 10$.

KEY
Set $x_1 = y_1 = 0$ in the distance formula. That gives the distance from the origin.

CHECK YOURSELF

9. The distance of a point (x, y) from the origin $O(0,0)$ is

- **A.** $x^2 + y^2$, leaving the square root off
- **B.** $\sqrt{x - y}$, subtracting instead of squaring and adding
- **C.** $\sqrt{x^2 + y^2}$

10. Why is the origin-distance formula just the distance formula with $x_1 = y_1 = 0$?

- **A.** the origin is just an ordinary point, $(0, 0)$
- **B.** because the origin is a special case that needs its own separate rule
- **C.** because distance from the origin is always smaller than distance between two other points
- **D.** because the origin lies on both axes at once, so it needs both formulas combined

1 more in this set online.

Naming a figure from its side lengths

Once we know every vertex of a triangle or a quadrilateral, the distance formula can name the figure. Nothing needs to be drawn, and nothing measured by eye.

Compute the length of every side. For a quadrilateral, compute both diagonals too. **Equal side lengths point to an isosceles or an equilateral triangle, or to a rhombus.** The pattern in the numbers alone reads off the shape.

A right angle needs one further check. Equal sides alone are never enough. Three side lengths confirm a right angle only through the converse of the Pythagoras theorem. The square of the longest side must equal the sum of the squares of the other two.

A square needs both conditions at once: all four sides equal, and both diagonals equal too. A rhombus can carry equal sides with unequal diagonals. Only a square carries both.

Two more names come off the same numbers, one condition at a time. Take the vertices in the order the question gives them, going round the figure, or the word OPPOSITE means nothing. If both pairs of opposite sides are equal, the figure is at least a PARALLELOGRAM. If the two diagonals are equal as well, it is a RECTANGLE. If all four sides are equal on top of that, it is a SQUARE.

A parallelogram whose diagonals differ has no right angle in it, and a parallelogram whose four sides are not all equal is not a rhombus.

Your turn: a quadrilateral has all sides 5, one diagonal 8 and the other 6. Is it a square? (Answer: no, it is a rhombus, since a square needs both diagonals equal.)

CAUTION
A right angle needs the Pythagoras converse on three sides. Equal sides alone do not show one.

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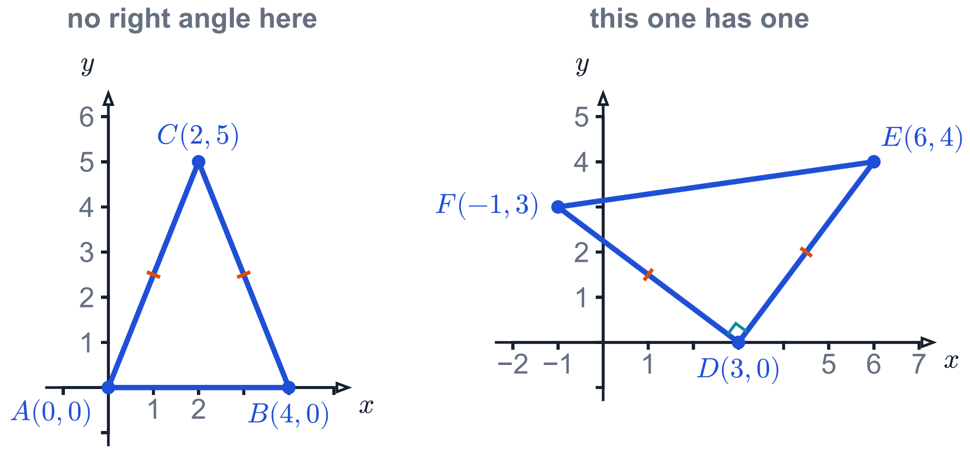
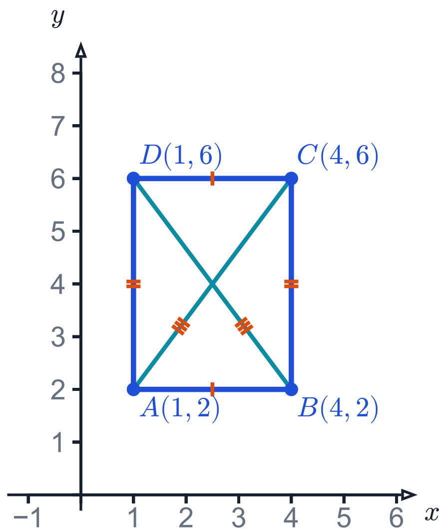
RULE
Equal sides alone give a rhombus; equal sides AND equal diagonals give a square.

Equal sides do not mean a right angle



Both triangles carry the same one-tick mark on their equal sides, but only one of the two has a right angle.

A rectangle needs a different check



The rectangle's two short sides carry one tick, its two long sides carry two ticks, and its diagonals carry three ticks.

CHECK YOURSELF

11. Given only the three side lengths of a triangle, how can you confirm it has a right angle without measuring any angle?

- **A.** check the converse Pythagoras theorem on the three sides
- **B.** check whether any two of the three sides are equal in length
- **C.** check whether the longest side is exactly twice the shortest
- **D.** draw the triangle to scale and measure the angle with a protractor

12. $A(0,0)$**,** $B(4,0)$**,** $C(0,3)$**. What type of triangle is** ABC**?**

- **A.** right-angled and scalene
- **B.** isosceles, since two of its sides look similar in length
- **C.** equilateral, since all three sides come from the same right triangle
- **D.** impossible to classify without also knowing the angles at B and C

2 more in this set online.

A point equidistant from two points

A point equidistant from two given points A and B sits at the same distance from each. That is an equation waiting to be written down.

We set the two distance expressions equal, then square both sides. Squaring removes the square root from each side. We expand the brackets, and the squared terms cancel each other out. What is left is a linear equation in the point's own coordinates.

Carry the expansion out in full before you collect anything. Stop halfway, and a square root stays tangled in the working, with nothing gained.

Put numbers on it. Which points (x, y) are the same distance from $A(1,2)$ as from $B(5,2)$? Set the two distances equal and square both sides at once.

$$(x-1)^2 + (y-2)^2 = (x-5)^2 + (y-2)^2$$

$$x^2 - 2x + 1 = x^2 - 10x + 25$$

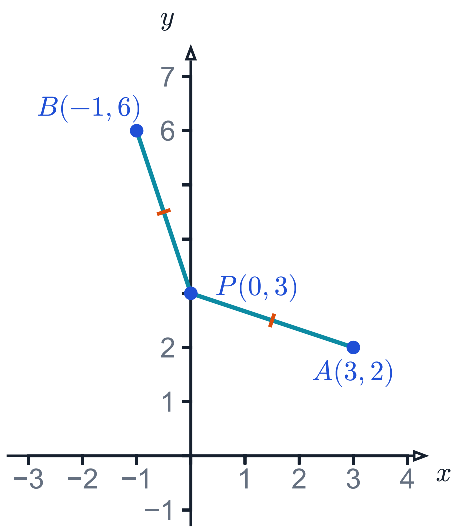
The x^2 terms cancel, and so does the whole $(y-2)^2$, which is why no y survives. What is left is $8x = 24$, so $x = 3$: every point on the one vertical line halfway between A and B.

Your turn: point $P(x, y)$ is equidistant from $A(0,0)$ and $B(4,0)$. Write down the equation you would square, before solving anything. (Answer: $\sqrt{x^2 + y^2} = \sqrt{(x-4)^2 + y^2}$, and squaring it leaves $x = 2$.)

RULE
Expand both squared expressions fully before collecting terms. The squared terms cancel, leaving a linear equation.

KEY
Equal distances, squared, removes both square roots at once.

Equal segments to two off-axis points



Point P on the y-axis is joined to points A and B by two segments, both tick-marked to show they share one length.

CHECK YOURSELF

13. A point equidistant from A and B is one whose

- **A.** distance from A plus its distance from B equals a fixed value
- **B.** coordinates are exactly the average of A's and B's coordinates
- **C.** lies exactly halfway between A and B along the segment AB
- **D.** distance from A equals its distance from B

14. Why does the method square both distance expressions before equating them, rather than equating the square roots directly?

- **A.** it removes the square roots, leaving a simple equation
- **B.** because square roots cannot be set equal to each other
- **C.** because squaring makes both distances positive, which they were not before
- **D.** because the two distances must first be converted to whole numbers

2 more in this set online.

The section formula for internal division

The point P divides the segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$. Its coordinates come from one formula: $((mx_2 + nx_1)/(m + n), (my_2 + ny_1)/(m + n))$.

Look closely at which letter multiplies which point's coordinates. **m pairs with B's coordinates, and n pairs with A's coordinates. That is mx_2, never mx_1.** We will use this exact pairing in every problem that follows. The ratio $m : n$ names A first and B second, but the formula's own pairing runs the other way round.

Try it yourself on a ratio of $2 : 3$ dividing A and B. It multiplies B's coordinates by 2, and A's coordinates by 3.

Your turn: a ratio $3 : 5$ divides the segment joining A and B. Which point's coordinates get multiplied by 3? (Answer: B's, since m always pairs with the second point.)

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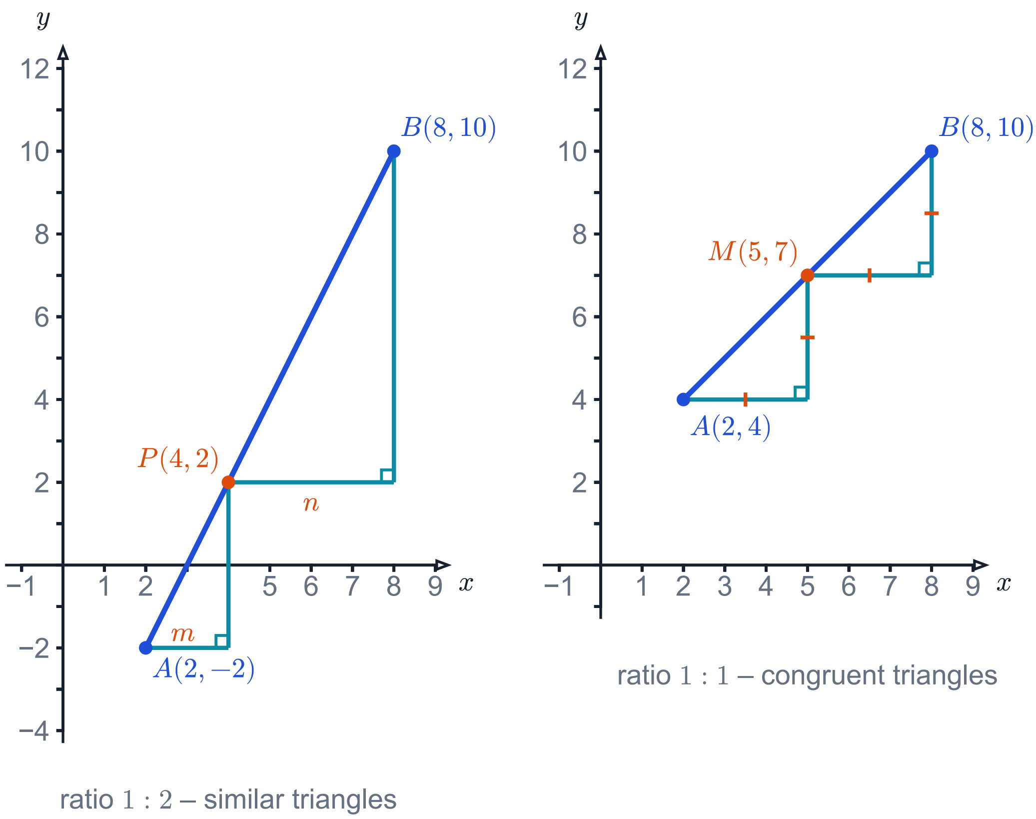
TRY
Ratio 2 : 3 dividing A, B: which coordinate does $m = 2$ multiply?

Answer: x_2 (and y_2), B's coordinates.

RULE
Match m to B's coordinates and n to A's coordinates; swapping either reverses the ratio.

The similar triangles behind the ratio



Two right-triangle constructions turn the ratio's numbers into actual leg lengths, once for a 1 to 2 division and once for the midpoint.

CHECK YOURSELF

15. P divides the segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$. P's coordinates are

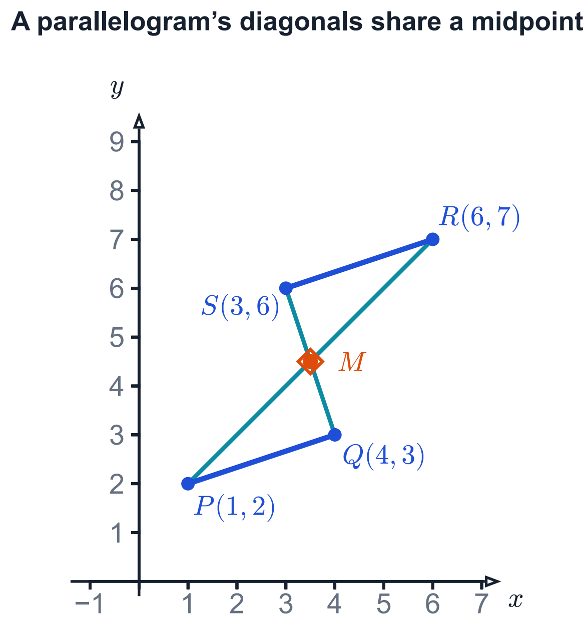
- **A.** $(mx_2 + nx_1, my_2 + ny_1)$, without dividing by $m + n$
- **B.** $((x_1 + x_2)/(m + n), (y_1 + y_2)/(m + n))$, an unweighted sum
- **C.** $((x_2 - x_1)/(m + n), (y_2 - y_1)/(m + n))$, borrowing a subtraction
- **D.** $((mx_2 + nx_1)/(m + n), (my_2 + ny_1)/(m + n))$

16. The section formula's coordinates are a weighted blend of A's and B's coordinates. What idea does that blend capture?

- **A.** P sits partway from A to B, at a fraction set by m and n
- **B.** the blend is only a shortcut for finding the midpoint, whatever m and n are
- **C.** the blend gives the point exactly halfway between A, B and the origin
- **D.** the blend gives the average of the two points' distances from the origin

3 more in this set online.

The midpoint formula



The parallelogram's diagonals cross at one shared point, which is the midpoint of each.

The midpoint of a segment is nothing new. It is the section formula's own special case, at the ratio $1:1$.

We set $m = n$ in the section formula. Both denominators become $m + n$, a number that cancels against itself either way. The midpoint works out to $((x_1 + x_2)/2, (y_1 + y_2)/2)$. Average the x's, then average the y's.

Try it on $(2, 4)$ and $(8, 10)$. You should get the midpoint $(5, 7)$, with no ratio bookkeeping required at all.

Your turn: find the midpoint of $(4, 2)$ and $(10, 8)$. (Answer: $(7, 5)$, since 4 and 10 average to 7, and 2 and 8 average to 5.)

TRY
Midpoint of $(2, 4)$ and $(8, 10)$?

Answer: $(5, 7)$.

RULE
The midpoint is the section formula's ratio $1:1$. Nothing new to memorise.

KEY
The midpoint is the one point with $AM = MB$, so equal distances are the check on your answer.

CHECK YOURSELF

17. The midpoint of the segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is

A. $(x_1 + x_2, y_1 + y_2)$

B. $((x_1 + x_2)/(m + n), (y_1 + y_2)/(m + n))$

C. $((x_1 + x_2)/2, (y_1 + y_2)/2)$

18. Why is the midpoint formula the special case of the section formula where $m = n$?

A. because a midpoint always lies at the exact centre of the coordinate plane

B. equal weights $m = n$ collapse the blend into a plain average

C. because $m = n$ forces both points to have the same coordinates

D. because the section formula only works at all when $m = n$

1 more in this set online.

Finding the ratio a point divides in

Given a point already known to lie on a segment, we can find the ratio it divides the segment in.

Write the unknown ratio as $k : 1$, so only one unknown remains. Substitute the point's x-coordinate into the section formula's x-half. Solve the resulting equation for k.

Checking the answer against the y-coordinate too is not optional. Both coordinates must agree on the same value of k. A mismatch means an arithmetic slip was made somewhere earlier: go back and find it.

Your turn: point $(2, 0)$ divides $(0, 0)$ and $(6, 0)$ in what ratio? (Answer: $1 : 2$, since $(2, 0)$ is a third of the way from $(0, 0)$ to $(6, 0)$.)

BACK
The section formula for internal division · p. 230

RULE
Check the answer against the OTHER coordinate too. Both must agree.

KEY
Write the unknown ratio as $k : 1$: one unknown, one equation, from either coordinate.

A SCHOLAR INDIA REMEMBERS

You solved for k from the x-coordinate. Now the y-coordinate gives a different k. Check the arithmetic once. If it is clean, the point was never on the segment. The second reading is telling you something. Do not argue with it. Correct the working, or correct the claim that the point lies on AB.

Brahmagupta · the brahminy kite

CHECK YOURSELF

19. Given a point known to lie on segment AB, the standard method for finding the ratio it divides AB in is to

- **A.** measure the segment with a ruler on a scale drawing
- **B.** always assume the ratio is $1 : 1$, since that is the most common case
- **C.** write the ratio as $k : 1$ and solve for k
- **D.** swap m and n from a ratio already used elsewhere in the chapter

20. Why does writing the ratio as $k : 1$ make it easier to solve for?

- **A.** it leaves only one unknown, k, to solve for
- **B.** it guarantees the ratio will always come out to a whole number
- **C.** it changes the section formula into the midpoint formula
- **D.** it removes the need to know the point's coordinates at all

2 more in this set online.

Common mistakes

Swapping the ratio's two numbers

Put n with x_2 and m with x_1 in the section formula. Surely the labels do not matter, since both numbers get used either way.

They do matter. Here is why. Take $A(0, 0)$ and $B(10, 0)$, divided in the ratio $1 : 4$. Here $m = 1$ and $n = 4$.

Done correctly, the formula gives $x = (mx_2 + nx_1)/(m+n)$. Substitute the numbers: $x = (1 \cdot 10 + 4 \cdot 0)/5 = 2$. That is the point $(2, 0)$, close to A, exactly as the smaller ratio number should place it.

Now swap the labels. Put n where m belongs, and m where n belongs. The formula gives $x = (4 \cdot 10 + 1 \cdot 0)/5 = 8$ instead, so the point becomes $(8, 0)$.

It is the correct point for the ratio $4 : 1$, not $1 : 4$. The two towns' roles have been reversed.

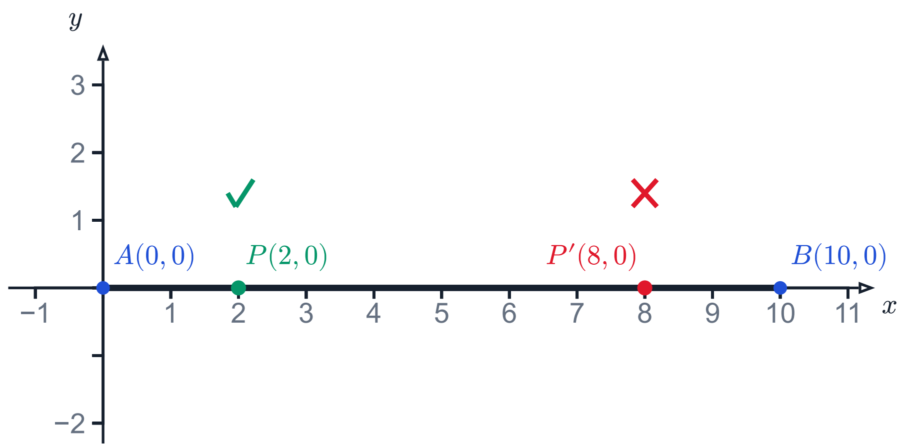
One safety check catches the slip every time. Does the point sit nearer the label with the smaller ratio number? $(2, 0)$ sits near A, matching the smaller ratio number, 1. $(8, 0)$ does not, and that alone tells us a swap happened somewhere in the working.

Your turn: $A(0, 0)$ and $B(20, 0)$ are divided in the ratio $1 : 3$. Is the dividing point $(5, 0)$ or $(15, 0)$? (Answer: $(5, 0)$, since $x = (1 \cdot 20 + 3 \cdot 0)/4 = 5$, near A.)

CAUTION
Match m to B, n to A. Swapping either one silently reverses which point the answer sits nearer to.

KEY
On $(2, -2)$ and $(8, 10)$: $1 : 2$ gives $(4, 2)$; $2 : 1$ gives $(6, 6)$. Swapping names a different point, not the same one.

Swap the labels, move the point



The point at 2, 0 sits near one end, and the swapped point at 8, 0 sits near the other end.

FIRST TRY
~~A is (0, 0) and B is (10, 0), ratio 1 : 4. I put the 4 with x_2 and the 1 with x_1, and got $x = (4 \cdot 10 + 1 \cdot 0)/5 = 8$.~~

SECOND LOOK
The ratio's first number multiplies the second point, B, and the second number multiplies the first point, A. So $m = 1$ pairs with $x_2 = 10$ and $n = 4$ pairs with $x_1 = 0$, giving $x = (1 \cdot 10 + 4 \cdot 0)/5 = 2$.

The ratio's first number always goes with the second point, B. Keep that order every time.

FIND THE POINT DIVIDING THE SEGMENT FROM A TO B IN THE RATIO 1 TO 4.

WEAK

Put n with x_2 and m with x_1. Surely it does not matter which label goes where.

For $A(0,0)$ and $B(10,0)$ in ratio $1:4$, that gives $x = (4 \cdot 10 + 1 \cdot 0)/5 = 8$, so the point comes out $(8,0)$.

But $(8,0)$ sits nearer B, not A. A point dividing in $1:4$ should sit nearer A, the end with the smaller number. Something has gone wrong.

STRONG

Put m with x_2 and n with x_1, matching the ratio's own order. For $A(0,0)$ and $B(10,0)$ in ratio $1:4$, $m = 1$ and $n = 4$, so $x = (1 \cdot 10 + 4 \cdot 0)/5 = 2$, giving the point $(2,0)$.

$(2,0)$ sits nearer A, matching the smaller ratio number, 1. This is the correct point for $1:4$.

CHECK YOURSELF

21. $A(0,0)$, $B(10,0)$, **ratio** $1:4$. **A student puts** n **with** x_2 **and** m **with** x_1, **and gets the point** $(8,0)$. **What went wrong?**

- **A.** nothing went wrong — $(8,0)$ is correct for the ratio $1:4$
- **B.** m and n were swapped
- **C.** the ratio should have been added instead of used as separate weights
- **D.** the y-coordinates were mixed up instead of the x-coordinates

22. $A(2,2)$, $B(12,7)$, **ratio** $2:3$. **A student computes the dividing point and gets** $(8,5)$. **What went wrong, if anything?**

- **A.** nothing went wrong — $(8,5)$ is correct
- **B.** the midpoint formula was used by mistake instead of the section formula
- **C.** m and n were swapped
- **D.** the coordinates of A and B were subtracted instead of combined with the ratio

3 more in this set online.

Worked examples

WORKED EXAMPLE · Distance between two points

1.	$\sqrt{(6-1)^2 + (14-2)^2}$	*Substitute the two points, $(1,2)$ and $(6,14)$, straight into the distance formula.*
2.	$= \sqrt{5^2 + 12^2} = \sqrt{25 + 144}$	*Subtract inside each bracket, then square each result.*
3.	$= \sqrt{169} = 13$	*Add the two squares, then take the square root.*
4.	$13^2 = 169$ and $5^2 + 12^2 = 25 + 144 = 169$	CHECK ***Check it by squaring the answer back.*** *It matches the sum of the two squared legs you started from, exactly.*

BACK

The distance between two points · p. 225

TRY

Find the distance between $(2,3)$ and $(5,7)$.

Answer: $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

RULE

Subtract, square, add, then take the square root: in that order.

THE DISTANCE BETWEEN TWO POINTS

1. **Label the two points** Call them (x_1, y_1) and (x_2, y_2). For $(1, 2)$ and $(6, 14)$, $x_1 = 1$, $y_1 = 2$, $x_2 = 6$, $y_2 = 14$.
2. **Subtract, then square** Find $x_2 - x_1$ and $y_2 - y_1$, then square each result. Here that is 5^2 and 12^2.
3. **Add, then root** Add the two squares, then take the square root. $25 + 144 = 169$, and $\sqrt{169} = 13$.
4. **Check the answer** Square the answer back. $13^2 = 169$, which matches the sum of the two squared legs.

CHECK YOURSELF

23. **Find the distance between $(2, 3)$ and $(10, 9)$.**

- A. 10
- B. $\sqrt{14}$
- C. 14
- D. 8

24. **A worked line reads: $\sqrt{(6-1)+(14-2)} = \sqrt{17}$, for the distance between $(1, 2)$ and $(6, 14)$. What is the correct fix?**

- A. take the square root of each difference separately, then add the two roots
- B. drop the square root entirely and just add the two differences
- C. square each difference before adding: $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$
- D. multiply the two differences together instead of adding them

1 more in this set online.

WORKED EXAMPLE · **Confirming a square by its sides**

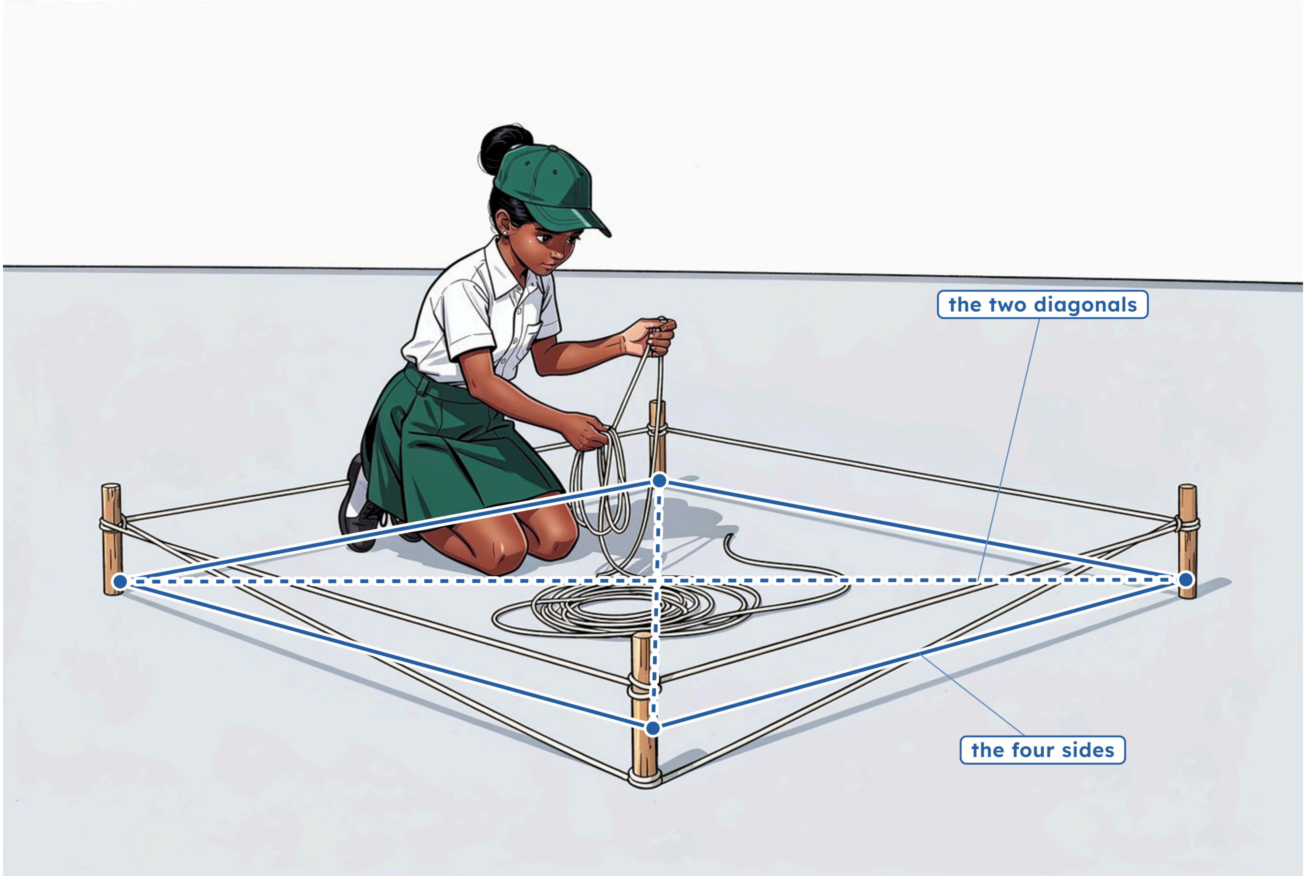
1.	$A(1,1)$, $B(4,5)$, $C(8,2)$, $D(5,-2)$	*List the four vertices before computing anything.*
2.	$AB = \sqrt{3^2 + 4^2} = 5$	*Apply the distance formula to the first side.*
3.	$BC = 5$, $CD = 5$, $DA = 5$	*The same formula applies to the remaining three sides. Every one comes out equal too.*
4.	$AC = \sqrt{7^2 + 1^2} = 5\sqrt{2}$, $BD = \sqrt{1^2 + 7^2} = 5\sqrt{2}$	*Compute both diagonals the same way. They too come out equal.*
5.	all four sides equal at 5, both diagonals equal at $5\sqrt{2}$	***Equal sides alone would only confirm a rhombus.*** *Equal diagonals as well are what confirm a square, since a parallelogram with equal diagonals has a right angle at every corner.*
6.	$AB^2 + BC^2 = 25 + 25 = 50 = \left(5\sqrt{2}\right)^2 = AC^2$	CHECK ***Check the right angle at B independently, using the Pythagoras converse.*** *The two squared sides add up to the squared diagonal, confirming the corner really is $90°$.*

BACK
Naming a figure from its side lengths · p. 227

TRY
Is triangle $A(0,0)$, $B(4,3)$, $C(8,0)$ isosceles? Compare AB and BC.

Answer: Yes: $AB = \sqrt{4^2+3^2} = 5$, $BC = \sqrt{4^2+3^2} = 5$; equal.

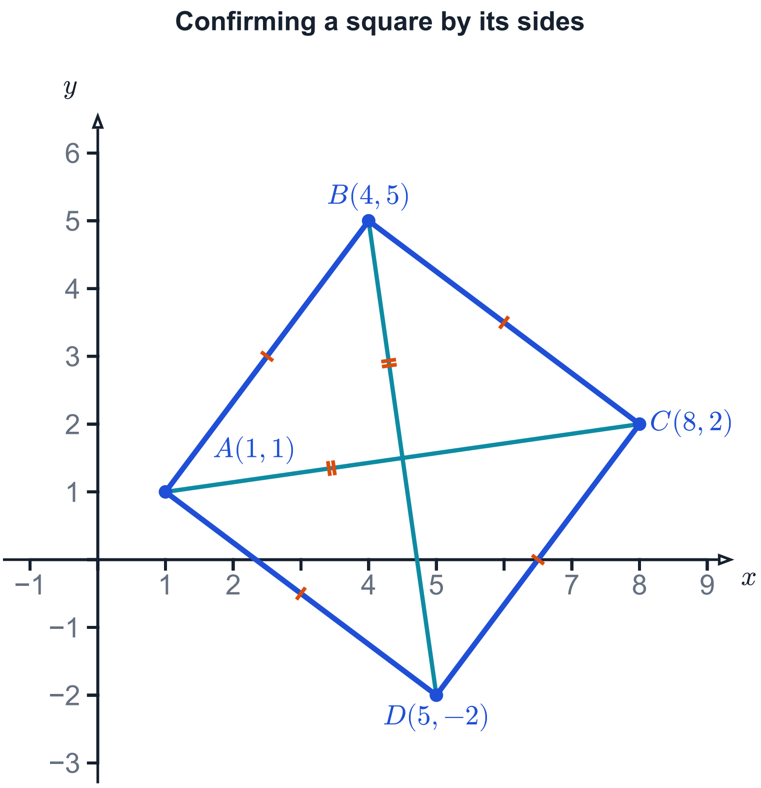
RULE
All four sides equal AND both diagonals equal is what separates a square from a rhombus.



Four pegs and lengths of string mark the quadrilateral Ananya is checking: the four sides, and both diagonals across it.

CONFIRMING A QUADRILATERAL IS A SQUARE

1. **List the vertices** Write down all four points before computing anything. Here that is $A(1,1)$, $B(4,5)$, $C(8,2)$, $D(5,-2)$.
2. **Measure every side** Apply the distance formula to all four sides in turn. Each one here comes out to 5.
3. **Measure both diagonals** Apply the distance formula to AC and BD too. Each one here comes out to $5\sqrt{2}$.
4. **Compare the lengths** Equal sides alone would only confirm a rhombus. Equal diagonals as well point to a square.
5. **Check the right angle** Use the Pythagoras converse on two sides and a diagonal at one corner. Here $5^2 + 5^2 = 50 = \left(5\sqrt{2}\right)^2$, so the corner really is 90°.



The quadrilateral shows four sides tick-marked with one equal length, and two diagonals tick-marked with a second, longer length.

CHECK YOURSELF

25. $A(0,0)$, $B(4,3)$, $C(8,0)$, $D(4,-3)$. **What shape do these four points form?**

- **A.** a square, since all four sides come out equal
- **B.** a rectangle, since opposite sides are parallel
- **C.** not a special quadrilateral, since the diagonals are different lengths
- **D.** a rhombus

26. The worked example confirmed $ABCD$ was a square only after checking the diagonals, not just the sides. Why was that extra check necessary?

- **A.** because the sides could have been measured incorrectly the first time
- **B.** because a square must have more than four equal sides
- **C.** because the diagonals must always be checked before the sides, never after
- **D.** to confirm the right angles a square needs, beyond a rhombus's equal sides

1 more in this set online.

WORKED EXAMPLE · **An equidistant point on an axis**

1.	let the point be $(0, y)$	*Any point on the y-axis has x-coordinate 0.*
2.	$9 + (2-y)^2 = 1 + (6-y)^2$	*Equate the squares of the distances to $A(3,2)$ and $B(-1,6)$ directly. This way, the square roots never appear.*
3.	$9 + 4 - 4y + y^2 = 1 + 36 - 12y + y^2$	*Expand both squared brackets fully.*

WORKED EXAMPLE · **An equidistant point on an axis**

4. $y = 3$

The y^2 terms cancel on both sides, leaving a linear equation to solve.

5. the point is $(0, 3)$, distance $\sqrt{10}$ from each

CHECK ***Substitute $y = 3$ back in. Check that both distances really do agree.***

BACK
A point equidistant from two points · p. 229

TRY
Find a point on the x-axis equidistant from $A(1, 3)$ and $B(5, 1)$.

Answer: $(2, 0)$. Equating squares, $(x-1)^2 + 9 = (x-5)^2 + 1$ gives $x = 2$.

RULE
A point on the y-axis is always $(0, y)$: one unknown, ready for the equal-distances equation.

A POINT ON AN AXIS EQUIDISTANT FROM TWO POINTS

1. Set up the unknown point A point on the y-axis is always $(0, y)$. Here we want it equidistant from $A(3, 2)$ and $B(-1, 6)$.

2. Equate the squares Set the two squared distances equal, so no square root appears. Here that is $9 + (2 - y)^2 = 1 + (6 - y)^2$.

3. Expand fully Expand both brackets before collecting anything. $9 + 4 - 4y + y^2 = 1 + 36 - 12y + y^2$.

4. Cancel and solve The y^2 terms cancel, leaving one linear equation. Solving it gives $y = 3$.

5. Check both distances Substitute $y = 3$ back into both distances. Both come out to $\sqrt{10}$, so the point $(0, 3)$ checks out.

CHECK YOURSELF

27. Find a point on the x-axis equidistant from $E(1, 5)$ and $F(4, -2)$.

- **A.** $(4, 0)$
- **B.** $(1, 0)$
- **C.** the midpoint of EF, at $(2.5, 1.5)$
- **D.** $(-4, 0)$

28. The worked example finds ONE point on an axis equidistant from A and B. Is that the only point in the plane equidistant from A and B?

- **A.** yes — it is the only equidistant point that exists
- **B.** no — the whole perpendicular bisector is equidistant
- **C.** yes, because equidistant points can only occur on one of the two axes
- **D.** no, but only two such points exist in total

1 more in this set online.

WORKED EXAMPLE · **Dividing a segment in a ratio**

1. $m = 1, n = 2$

Read the ratio $1 : 2$ directly as m and n, for $A(2, -2)$ and $B(8, 10)$.

WORKED EXAMPLE · **Dividing a segment in a ratio**

2.	$x = (1 \cdot 8 + 2 \cdot 2)/3 = 4$	*Substitute into the section formula's x-half. m multiplies B's x-coordinate, and n multiplies A's.*
3.	$y = (1 \cdot 10 + 2 \cdot (-2))/3 = 2$	*The same substitution on the y-half.*
4.	the point is $(4, 2)$	*Pair the two coordinates you just found.*
5.	$AP = \sqrt{20} = 2\sqrt{5}$ and $PB = \sqrt{80} = 4\sqrt{5}$	CHECK ***Check the ratio by measuring both pieces with the distance formula.*** *They come out $2\sqrt{5} : 4\sqrt{5} = 1 : 2$, exactly as given.*

BACK
The section formula for internal division · p. 230

TRY
Find the point dividing $A(1, 2)$, $B(7, 8)$ in ratio $1 : 2$.

Answer: $(3, 4)$: $x = (1 \cdot 7 + 2 \cdot 1)/3 = 3$, $y = (1 \cdot 8 + 2 \cdot 2)/3 = 4$.

RULE
Substitute m and n into the formula directly. No need to solve for anything.

DIVIDING A SEGMENT IN A GIVEN RATIO

1. **Read off m and n** Match the ratio's two numbers to m and n in order. Here the ratio $1 : 2$ gives $m = 1$, $n = 2$, for $A(2, -2)$ and $B(8, 10)$.
2. **Substitute into the x-half** m multiplies B's x-coordinate, n multiplies A's. Here $x = (1 \cdot 8 + 2 \cdot 2)/3 = 4$.
3. **Substitute into the y-half** Use the same pairing on the y-coordinates. Here $y = (1 \cdot 10 + 2 \cdot (-2))/3 = 2$.
4. **Pair and check** The point is $(4, 2)$. Check it by measuring both pieces with the distance formula, and confirm the ratio.

CHECK YOURSELF

29. Find the point dividing the segment joining $A(-4, -1)$ and $B(6, 9)$ internally in the ratio $3 : 2$.

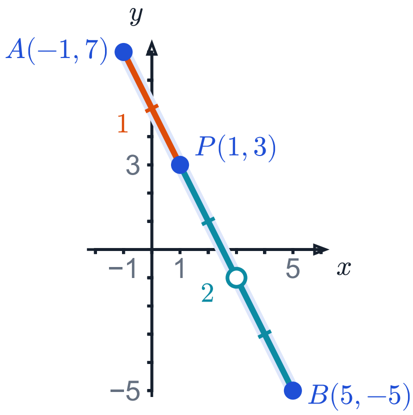
- **A.** $(0, 3)$
- **B.** $(1, 4)$
- **C.** $(2, 5)$
- **D.** $(-2, 3)$

30. In the worked example, $A(2, -2)$, $B(8, 10)$, ratio $1 : 2$ gives the point $(4, 2)$, which sits closer to A than to B. Why does that make sense?

- **A.** $AP : PB = 1 : 2$ makes AP the smaller share, nearer A
- **B.** it does not make sense — the point should sit closer to B instead
- **C.** it is just a coincidence for this particular pair of points
- **D.** it only makes sense because A has negative coordinates

1 more in this set online.

The known point fixes the ratio



The known point cuts AB into three equal pieces – one on the A side of it, two on the B side.

Here the point is the given and the ratio is the unknown, the reverse of the example before it.

WORKED EXAMPLE · **Finding an unknown ratio**

1.	write the ratio as $k:1$	*One unknown ratio needs only one unknown, for $A(-1,7)$, $B(5,-5)$ and the known point $(1,3)$.*
2.	$1=(5k-1)/(k+1)$	*Substitute the known point's x-coordinate into the section formula's x-half.*
3.	$5k-1=k+1$, so $k=1/2$	*Clear the denominator, then collect terms and solve.*
4.	the ratio is $1:2$	*$k=1/2$ written as a ratio to 1.*
5.	checked against the y-coordinate: $y=(1/2\cdot(-5)+1\cdot 7)/(1/2+1)=3$	CHECK ***The same k must satisfy both coordinates.*** *This confirms no arithmetic slip was made.*

BACK
Finding the ratio a point divides in · p. 233

TRY
Point $(6,7)$ lies on the segment joining $A(2,1)$ and $B(8,10)$. Find the ratio.

Answer: $2:1$. Solving $6=(8k+2)/(k+1)$ gives $k=2$.

RULE
Solving from the x-coordinate is enough. The y-coordinate is only there to check the answer.

FINDING THE RATIO A POINT DIVIDES A SEGMENT IN

1. **Write the ratio** Write the unknown ratio as $k:1$. One unknown needs only one unknown, for $A(-1,7)$, $B(5,-5)$ and the known point $(1,3)$.
2. **Substitute the x-coordinate** Put the known point's x-coordinate into the section formula's x-half. Here $1=(5k-1)/(k+1)$.
3. **Solve for k** Clear the denominator, then collect terms. Here $5k-1=k+1$, so $k=1/2$.
4. **State the ratio** Turn k back into a ratio to 1. Here $k=1/2$ gives the ratio $1:2$.
5. **Check with the other coordinate** The same k must satisfy the y-coordinate too. Here it does, confirming no arithmetic slip was made.

CHECK YOURSELF

31. Point $(2,-3)$ lies on the segment joining $A(-4,3)$ and $B(5,-6)$. Find the ratio it divides the segment in.

- **A.** $1:2$
- **B.** $2:3$
- **C.** $2:1$
- **D.** $3:2$

32. The worked example checks the ratio using both the x-coordinate equation AND the y-coordinate equation. Why check both, instead of stopping after the first?

- **A.** because the x-coordinate equation is never reliable on its own
- **B.** because CBSE marking always requires two separate equations to be shown
- **C.** it confirms the point truly lies on the segment, not by coincidence
- **D.** because the y-coordinate always gives a different value of k than the x-coordinate does

1 more in this set online.

Wrapping up

IN EVERY CHAPTER

Recap

- DISTANCE: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. For $(1,2)$ and $(6,14)$, $d=13$.
- FROM ORIGIN: $d=\sqrt{x^2+y^2}$. For $(6,8)$, $d=10$.
- SECTION: divide $(2,-2)$ and $(8,10)$ in ratio $1:2$, and we land at $(4,2)$.
- MIDPOINT: the ratio $1:1$ case. Average $(2,4)$ and $(8,10)$, and we get $(5,7)$.
- LABEL ORDER: m pairs with the second point, and n with the first. Swap them, and $(2,0)$ becomes $(8,0)$: nearer B, when the smaller ratio number should keep it nearer A.

Look back over these five lines. Only two formulas are doing all the work.

The section formula works the other way round. Given a ratio, it finds the point. Given the point already found, the same formula finds the ratio back. The midpoint is its simplest case, the ratio $1:1$.

Your turn: find the midpoint of $(0,2)$ and $(8,4)$, then its distance from the origin. (Answer: midpoint $(4,3)$, distance 5, since $\sqrt{4^2+3^2}=\sqrt{25}=5$.)

And Kabir's question from the opening page. The two discs sit at $(3,2)$ and $(7,6)$, so one is 4 tiles across and 4 tiles up from the other, and the distance between them is $\sqrt{4^2+4^2}=\sqrt{32}=4\sqrt{2}$, about 5.7 tiles. Four numbers in, one length out.

KEY
Two formulas: distance turns two points into a length, section turns a ratio into a point.

BACK
The distance between two points · p. 225

CHECK YOURSELF

33. Two mobile towers stand at known coordinates, and an engineer wants to know how far apart they are. Which formula applies?

A. the section formula

B. the midpoint formula

C. the distance formula

D. no coordinate formula — this needs calculus instead

34. A road planner wants a rest stop $2/5$ of the way from town A to town B along a straight road. Which formula locates it?

A. the distance formula

B. the midpoint formula

C. the section formula, using the ratio $2:3$

D. the average of the two towns' distances from the origin

1 more in this set online.

CHECK YOURSELF: THE WHOLE CHAPTER

35. According to this chapter's opening claim, the distance between the points (x_1, y_1) and (x_2, y_2) is given by which expression?

A. $(x_2 - x_1) + (y_2 - y_1)$

B. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

C. $\sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$

D. $(x_2 - x_1)^2 + (y_2 - y_1)^2$

36. What is the distance between the points $(3, 2)$ and $(7, 5)$?

A. 7

B. $\sqrt{74}$

C. 5

D. $\sqrt{7}$

3 more in this set online.

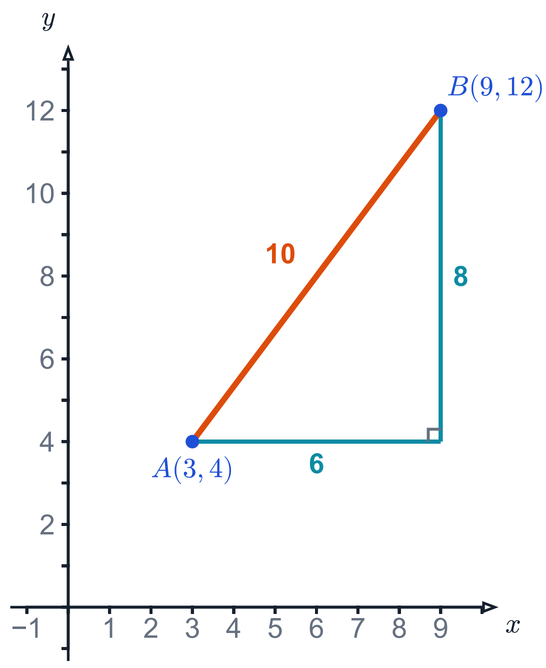
IN EVERY CHAPTER

Where you will meet this

You will use coordinates whenever a length or a matching point comes from two locations on a grid. Here are seven of those places.

- **Marking two points of a rangoli design.** A rangoli design has two marked points, $A(3, 4)$ and $B(9, 12)$, on graph paper.
  The maths: the distance formula turns two points into one length. It uses the horizontal and vertical gap between them.
  The horizontal gap is $9 - 3 = 6$. The vertical gap is $12 - 4 = 8$, so the distance is $\sqrt{6^2 + 8^2} = 10$.

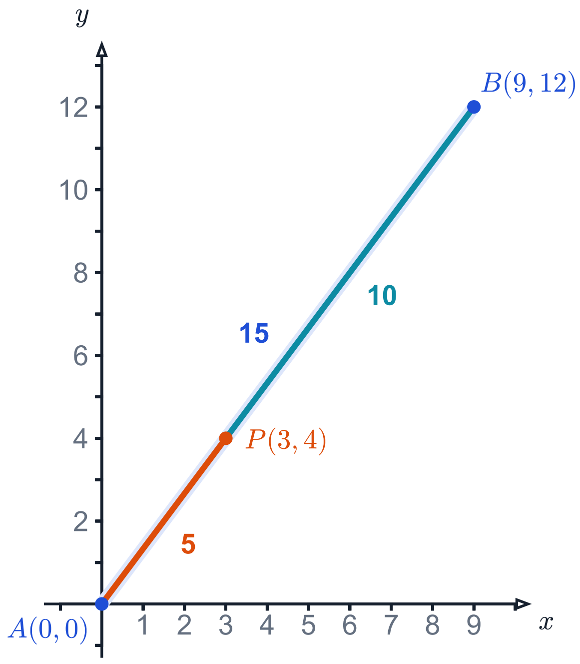
The rangoli points, six across and eight up



The gap across is 6 and the gap up is 8; they are the two legs, and the distance from A to B is the hypotenuse, 10.

Two points on a grid are the ends of a right triangle whose legs are the horizontal and vertical gaps.

The cone a third of the way



P is a third of the way from A to B, 5 of the 15 metres, so the two pieces, 5 and 10, read the ratio one to two.

Point P divides the 15 m segment from A to B in the ratio one to two, landing 5 m from A.

- **Checking a patchwork block is square.** A quilter marks four corners of a patchwork block on grid paper: $A(0,0)$, $B(5,12)$, $C(17,7)$, $D(12,-5)$.
  The maths: compute every side and both diagonals with the distance formula. Equal sides and equal diagonals confirm a square.

Every side comes out to 13. Both diagonals come out to $13\sqrt{2}$, so all four corners form a square.

- **Reading a savings tracker's halfway point.** A savings tracker plots two check-ins as points. $P(4, 800)$ and $Q(10, 2000)$ show week against balance in rupees.
  The maths: the midpoint formula averages the two coordinates, to estimate the balance halfway between the two check-ins.
  The midpoint is $((4 + 10)/2, (800 + 2000)/2) = (7, 1400)$, around week 7 at about ₹1400.
- **Placing a practice cone at a fixed ratio.** A coach marks two cones, $A(0, 0)$ and $B(9, 12)$, 15 m apart on the field.
  The maths: the section formula finds a point that divides a segment in a given ratio, without measuring anything by hand.
  Dividing AB in ratio $1 : 2$ from A gives $((1 \cdot 9 + 2 \cdot 0)/3, (1 \cdot 12 + 2 \cdot 0)/3) = (3, 4)$, a new cone 5 m from A.
- **Finding a bus stop's distance from the centre.** A town map puts the town centre at the origin, $O(0, 0)$, one grid unit to 100 m. A new bus stop sits at $S(8, 15)$.
  The maths: the distance from the origin is the same distance formula, with one point fixed at the origin.
  The distance is $\sqrt{8^2 + 15^2} = 17$ units, which is 1.7 km.
- **Placing a cutting table equally far from two doors.** A tailor's shop has two doors, $A(5, 4)$ and $B(-3, 8)$. The back wall is the line $x = 0$. A cutting table must sit on it, equally far from both doors.
  The maths: a point equidistant from two points makes their squared distances equal. Solving for the unknown coordinate finds the point.
  Squaring both distances and solving gives $y = 4$, so the table sits at $(0, 4)$, distance 5 from each door.
- **Placing a water station between two stalls.** An event planner's spreadsheet marks two food stalls, $A(2, 3)$ and $B(17, 18)$. A water station must divide AB in the ratio $2 : 3$.
  The maths: the spreadsheet must pair each ratio number with the correct point. Swapping the two numbers sends the station to the wrong spot.
  Correctly, the station lands at $(8, 9)$. Swapped, it lands at $(11, 12)$, closer to B instead of A.

Your turn. A garden path has two lamp posts, $A(2, 3)$ and $B(10, 9)$. Find the distance between them, and the point exactly halfway. *Answer:* The distance is $\sqrt{8^2 + 6^2} = 10$ m. The halfway point, the midpoint, is $((2 + 10)/2, (3 + 9)/2) = (6, 6)$.

Practice: Exercise 7.1

PRACTICE SET

1. PRACTICE Find the distance between $(3, 1)$ and $(7, 4)$. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	*write the distance formula down before any number goes into it*
2.	$x_2 - x_1 = 7 - 3 = 4$ and $y_2 - y_1 = 4 - 1 = 3$	*take the two differences on their own, so a sign mistake shows up here and not later*
3.	$d = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25}$	*square each difference, which makes both terms positive whichever way round you subtracted*
4.	$d = 5$	*the check — the two differences are 4 and 3, the legs of a 3-4-5 right triangle, so a whole-number answer is expected here*

2. PRACTICE Find the distance of the point $(-9, 40)$ from the origin. (Start the same way — the origin is the point $(0, 0)$, so put that in as the second point.)

3. PRACTICE Show that $A(1, 1)$, $B(5, 1)$ and $C(3, 5)$ are the vertices of an isosceles triangle. (Find all three side lengths with the same formula, then compare them.)

4. PRACTICE Find the distance between $(2, 3)$ and $(4, 1)$.

5. PRACTICE Find the distance of the point $(-5, 12)$ from the origin.

6. PRACTICE Two towns sit at $(0, 0)$ and $(24, 7)$ on a district grid, one grid unit to a kilometre. Find the straight-line distance between them.

7. PRACTICE Show that $(3, 0)$, $(6, 4)$ and $(-1, 3)$ are the vertices of an isosceles right triangle.

8. PRACTICE Show that $(1, 2)$, $(4, 2)$, $(4, 6)$ and $(1, 6)$ are the vertices of a rectangle.

9. PRACTICE Find a point on the x-axis equidistant from $(7, 6)$ and $(-3, 4)$.

ANSWERS **1.** $d = 5$. **2.** $d = 41$, since $\sqrt{(-9)^2 + 40^2} = \sqrt{81 + 1600} = \sqrt{1681}$.
3. $AB = 4$, $BC = \sqrt{20}$ and $CA = \sqrt{20}$. Two sides are equal, so the triangle is isosceles.
4. $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. **5.** $\sqrt{(-5)^2 + 12^2} = 13$. **6.** $\sqrt{24^2 + 7^2} = 25$ km.
7. The three sides are 5, 5 and $5\sqrt{2}$; $5^2 + 5^2 = \left(5\sqrt{2}\right)^2$, confirming a right angle where the two equal sides meet.
8. Opposite sides come out 3 and 4, and both diagonals come out 5, confirming a rectangle.
9. $(3, 0)$.
Full solutions: **manthika.com/learn/math10-ch07**

Further practice

PRACTICE SET

10. PRACTICE Find the distance between $(-5, 7)$ and $(-1, 4)$.

SOLUTION — Q10 · **Worked in full**

1.	$d^2 = (-1 - (-5))^2 + (4 - 7)^2$	*The distance formula, squared, on the two given points.*
2.	$d^2 = 16 + 9 = 25$	*Square each difference and add.*
3.	$d = \sqrt{25} = 5$	*Take the square root.*

11. PRACTICE Which of these is the distance of the point $(8, 15)$ from the origin?

A. 17
B. 23
C. $\sqrt{161}$
D. 15

12. PRACTICE Two water tanks on a farm map sit at $(0, 0)$ and $(9, 12)$, one grid unit to a metre. Find the distance between them.

13. PRACTICE Show that $A(0, 0)$, $B(4, 0)$ and $C(2, 5)$ are the vertices of an isosceles triangle.

14. PRACTICE Show that $A(4, -2)$, $B(7, 1)$ and $C(13, 7)$ are collinear.

15. PRACTICE Show that $A(0, 0)$, $B(3, 4)$, $C(0, 8)$ and $D(-3, 4)$, taken in order, are the vertices of a rhombus but not a square.

16. PRACTICE Show that $A(0, 0)$, $B(4, 0)$, $C(6, 3)$ and $D(2, 3)$, taken in order, are the vertices of a parallelogram that is not a rhombus.

17. PRACTICE Find a point on the y-axis equidistant from $A(3, -2)$ and $B(-5, 6)$.

18. PRACTICE Find the values of y for which the distance between $(1, 2)$ and $(4, y)$ is 5.

19. PRACTICE Find k such that the point $(k, 3)$ is equidistant from $A(5, -2)$ and $B(1, 4)$.

ANSWERS **10.** 5 **11.** A — 17. **12.** 15 m.
13. $AC = BC = \sqrt{29}$ and $AB = 4$; two sides are equal, so the triangle is isosceles.
14. $AB = 3\sqrt{2}$, $BC = 6\sqrt{2}$ and $AC = 9\sqrt{2}$; since $AB + BC = AC$, the three points are collinear.
15. Every side works out to 5, but the diagonals $AC = 8$ and $BD = 6$ are not equal; equal sides with unequal diagonals give a rhombus, not a square.
16. $AB = CD = 4$ and $BC = DA = \sqrt{13}$; opposite sides are equal, giving a parallelogram, but all four sides are not equal, so it is not a rhombus.
17. $(0, 3)$ **18.** $y = 6$ or $y = -2$ **19.** $k = 6$
Full solutions: **manthika.com/learn/math10-ch07**

Practice: Exercise 7.2

PRACTICE SET

1. PRACTICE Find the point that divides the segment joining $(1, 2)$ and $(7, 5)$ internally in the ratio $2 : 1$. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1.	$P = ((mx_2 + nx_1)/(m + n), (my_2 + ny_1)/(m + n))$	*write the section formula out, and label which point is which before substituting:* $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (7, 5)$, $m : n = 2 : 1$
2.	$x = (2 \cdot 7 + 1 \cdot 1)/(2 + 1) = 15/3 = 5$	m *multiplies the SECOND point and* n *the first — swapping them gives the point that divides in the ratio* $1 : 2$ *instead, which is a different point*
3.	$y = (2 \cdot 5 + 1 \cdot 2)/(2 + 1) = 12/3 = 4$	*the* y *coordinate is built the same way, with the same* m *and* n *in the same places*
4.	$P = (5, 4)$	*the check —* P *should be twice as far from* $(1, 2)$ *as from* $(7, 5)$*, and it is:* $\sqrt{20}$ *against* $\sqrt{5}$

2. PRACTICE Find the midpoint of the segment joining $(-3, 8)$ and $(5, -2)$. (The midpoint is the section formula with the two parts equal, so the halves cancel to a simple average.)

3. PRACTICE In what ratio does the point $(3, 0)$ divide the segment joining $(1, -2)$ and $(4, 1)$? (Start the same way, but here the point is known and the ratio is not — write the unknown ratio as $k : 1$, the way the chapter does, so only one unknown is left.)

4. PRACTICE Find the coordinates of the point dividing the segment joining $(4, -1)$ and $(-2, -3)$ internally in the ratio $1 : 2$.

5. PRACTICE Find the midpoint of the segment joining $(6, -5)$ and $(-2, 11)$.

6. PRACTICE Three vertices of a parallelogram $PQRS$ are $P(1, 2)$, $Q(4, 3)$ and $R(6, 7)$. Find the fourth vertex S, using the fact that a parallelogram's diagonals share their midpoint.

7. PRACTICE The point $(0, y)$ divides the segment joining $(-4, 6)$ and $(6, -4)$ internally. Find y and the ratio of division.

ANSWERS **1.** $P = (5, 4)$. **2.** $(1, 3)$.
3. $2 : 1$. Write the unknown ratio as $k : 1$, so only one unknown is left. The x coordinate gives $3 = (4k + 1)/(k + 1)$, so $3k + 3 = 4k + 1$ and $k = 2$, which is the ratio $2 : 1$. Check it against the y coordinate: $(2 \cdot 1 + 1 \cdot (-2))/(2 + 1) = 0$, which agrees.
4. $(2, -5/3)$. **5.** $(2, 3)$.
6. $S(3, 6)$, from equating the midpoint of PR with the midpoint of QS.
7. Ratio $2 : 3$, $y = 2$.
Full solutions: **manthika.com/learn/math10-ch07**

Further practice

PRACTICE SET

8. PRACTICE Find the point dividing the segment joining $A(-4, 3)$ and $B(6, -2)$ internally in the ratio $3 : 2$.

SOLUTION — Q8 · **Worked in full**

1.	$x = (3 \cdot 6 + 2 \cdot (-4))/(3 + 2)$	*Substitute into the section formula's x-half — m multiplies B's x-coordinate, n multiplies A's.*
2.	$x = (18 - 8)/5 = 2$	*Simplify the numerator, then divide.*
3.	$y = (3 \cdot (-2) + 2 \cdot 3)/(3 + 2) = (-6 + 6)/5 = 0$	*The same substitution on the y-half.*

9. PRACTICE Find the midpoint of the segment joining $(-3, 5)$ and $(9, -1)$.

10. PRACTICE Which of these is the point dividing the segment joining $(0, 0)$ and $(10, 15)$ internally in the ratio $2 : 3$?

A. $(4, 6)$
B. $(6, 9)$
C. $(2, 3)$
D. $(8, 12)$

11. PRACTICE Which of these is the midpoint of the segment joining $(-6, 3)$ and $(2, -9)$?

A. $(-2, -3)$
B. $(-4, 6)$
C. $(4, -3)$
D. $(-2, 3)$

12. PRACTICE Find the coordinates of the points of trisection of the segment joining $A(2, -4)$ and $B(-4, 8)$.

13. PRACTICE Three vertices of a parallelogram $PQRS$, taken in order, are $P(1, -2)$, $Q(4, 0)$ and $R(7, 6)$. Find the fourth vertex S, using the fact that a parallelogram's diagonals share their midpoint.

14. PRACTICE In what ratio does the x-axis divide the segment joining $A(1, 4)$ and $B(6, -6)$? Also find the point of division.

15. PRACTICE Find the ratio in which the point $(0, -1)$ divides the segment joining $A(3, 5)$ and $B(-2, -5)$.

16. PRACTICE P lies on the segment joining $A(1, -2)$ and $B(9, 6)$ such that $AP = 3/8 \cdot AB$. Find the coordinates of P.

17. PRACTICE The point $(a, 1)$ divides the segment joining $(2, -5)$ and $(-4, 4)$ internally in the ratio $2 : 1$. Find a.

ANSWERS **8.** $(2, 0)$ **9.** $(3, 2)$ **10.** A — $(4, 6)$. **11.** A — $(-2, -3)$. **12.** $P(0, 0)$ and $Q(-2, 4)$. **13.** $S(4, 4)$ **14.** Ratio $2 : 3$, point $(3, 0)$. **15.** Ratio $3 : 2$. **16.** $P(4, 1)$ **17.** $a = -2$
Full solutions: **manthika.com/learn/math10-ch07**