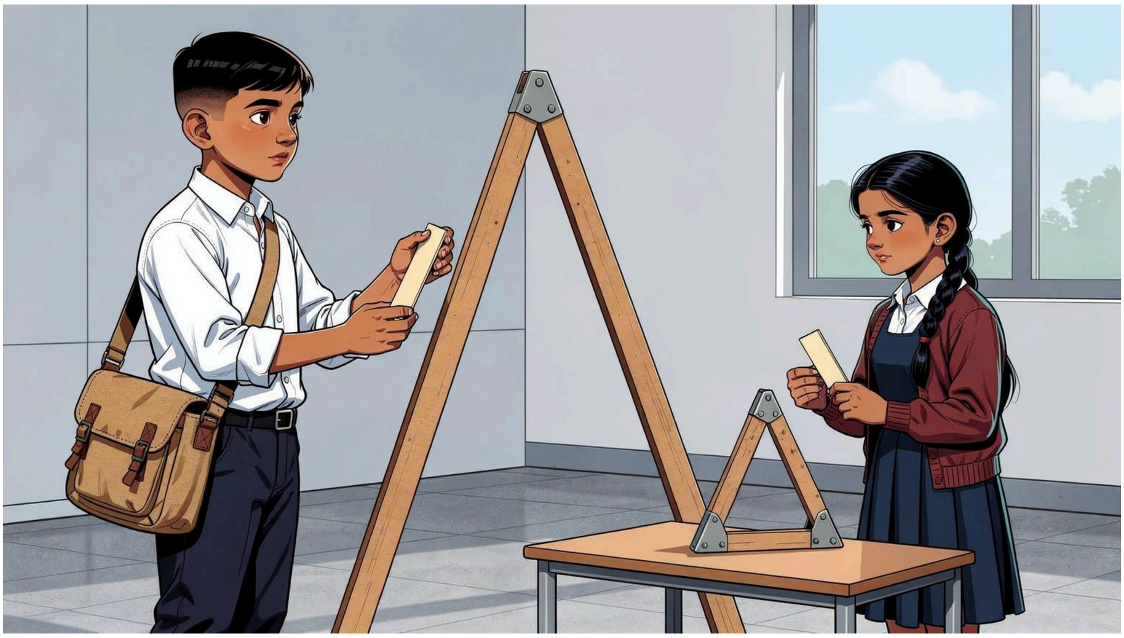
8 CHAPTER 8 Introduction to Trigonometry

WHAT THIS CHAPTER BUILDS
You will learn the six ratios of an acute angle, their exact values at the standard angles, and the one identity that lets any ratio be rewritten in terms of another.

TRIGONOMETRY UNIT, BOARD CORE

IN THIS CHAPTER



Aarav and Meera compare a tall and a small wooden triangle cut to the same angle, checking their sides keep the same ratio.

Getting started

Ratios of a right triangle

A STORY

Two frames, same angle

Aarav and Meera stand in the maths lab with two wooden frames. Both are cut to the same sharp angle. One frame is much taller than the other.

Aarav holds a plain wooden strip up beside the tall frame. Meera holds a strip just like it beside the small frame, which stands on the table next to her.

“Same angle, different size,” Meera says. “My frame is a lot smaller than yours.”

“Suppose a triangle has a square corner,” Aarav says. “And I know one other angle and one side.”

Meera looks at her own small frame. “Then you can find the other two sides,” she says. “In a right triangle, three ratios of the sides depend only on that angle. Not on how big it is.”

“Only on the angle?” Aarav asks. “Not the size at all?”

“Not the size at all,” Meera says. “A tiny right triangle and a huge one, same angle, same three ratios.”

Could the same three ratios really hold for a right triangle ten times as tall?

Pick an acute angle inside a right triangle, and call it A. Three ratios of the triangle’s sides depend on A alone, never on the size of the triangle.

Draw a right triangle and mark angle A. Look at the side opposite A, and the hypotenuse. Divide one by the other. Draw the triangle bigger or smaller. The ratio stays the same number, for that one value of A. This ratio has a name, sine, written $\sin A$. A second ratio, cosine, is $\cos A$. A third, tangent, is $\tan A$. Both are built from the triangle’s other two sides.

The Baudhayana Sulba-sutra, an Indian altar-construction text from about 800 BCE, already used this relation to lay out altars. Centuries later, Aryabhata’s Aryabhatiya (499 CE) tabulated sine values like these, to compute planetary positions.

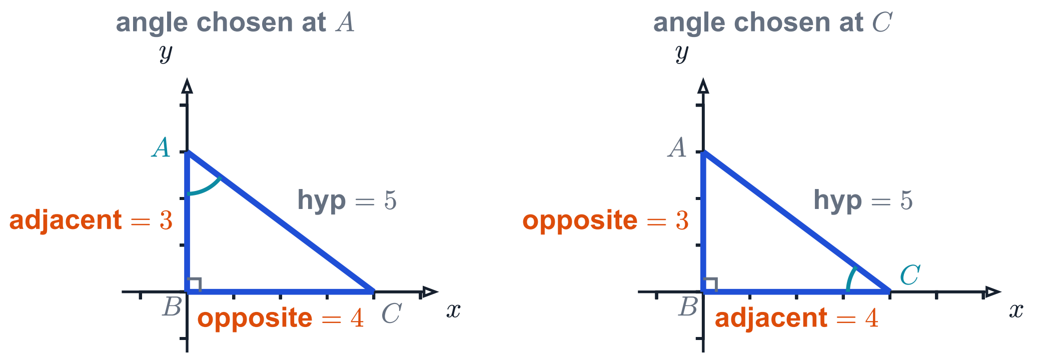
Five angles get exact values: $0°$, $30°$, $45°$, $60°$ and $90°$. One identity, $\sin^2 A + \cos^2 A = 1$, ties $\sin A$ and $\cos A$ together for every acute angle. We prove both of these in full, not just state them.

Your turn: for a 3-4-5 triangle, does $\sin^2 A + \cos^2 A$ come out to 1? (Answer: yes. $(3/5)^2 + (4/5)^2 = 9/25 + 16/25 = 1$.)

RULE
Before using any ratio, draw the right triangle and mark opposite, adjacent and hypotenuse against angle A.

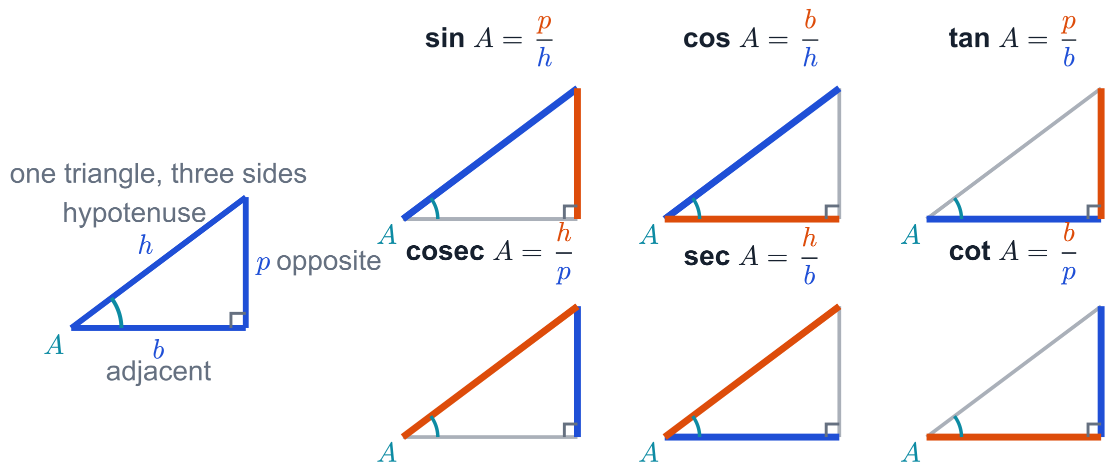
KEY
This chapter turns an angle into six ratios of a right triangle’s sides, and shows the ratios fix the angle back again.

One triangle, two angles, the sides swap names



Picking the angle at vertex A names one side opposite and one adjacent; picking vertex C instead swaps both labels.

Six names, two sides each



One triangle has three sides, and each of the six names picks two of them and writes one over the other.

IN THE WILD

A short ladder leaning at 60 degrees and a crane's long arm at the same angle have very different lengths. Height divided by length still gives the same number for both, because sine, cosine and tangent depend only on the angle, never on the triangle's size. That is why one small table of values, at 30, 45 and 60 degrees, serves every right triangle there is.

CHECK YOURSELF

1. **In this chapter, the ratios** $\sin A$, $\cos A$ **and** $\tan A$ **for an acute angle** A **of a right triangle depend on**

- **A.** the angle A alone, never the size of the triangle
- **B.** the size of the triangle, since a larger triangle gives larger ratios
- **C.** which side the triangle-drawer chooses to call the hypotenuse
- **D.** whether the sides are measured in centimetres or metres

2. Suppose $\sin A$ **really did change with the size of the right triangle carrying angle** A**. What would that break?**

- **A.** one angle would no longer give one fixed value of $\sin A$
- **B.** nothing would break, since every right triangle still has a hypotenuse
- **C.** Pythagoras' theorem would stop holding for that triangle
- **D.** the triangle would no longer be a right triangle

1 more in this set online.

IN EVERY CHAPTER

Before you start

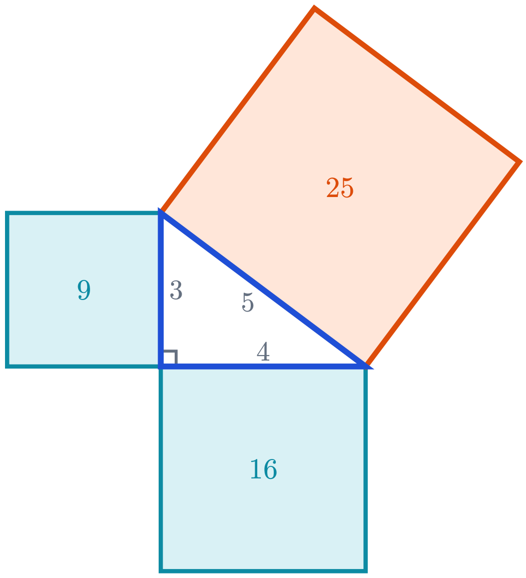
DO THIS FIRST

You already know the shapes and the numbers behind this. Now put them together, one ratio at a time. Try each check below.

- **Pythagoras' theorem** (Class 8, Ganita Prakash Part 2, page 44).
  Here: the identity $\sin^2 A + \cos^2 A = 1$ needs $p^2 + b^2 = h^2$.
  Check: $3^2 + 4^2 = 5^2$, since $9 + 16 = 25$.
- **A ratio in simplest form** (Class 8, Ganita Prakash Part 1, page 161).
  Here: sides $4k$ and $3k$ use a ratio already in this form.
  Check: $4 : 3$ has no common factor, so it is already in simplest form.
- **Undoing a square** (Class 8, Ganita Prakash Part 1, page 8).
  Here: finding a third side, $\sqrt{25k^2} = 5k$, undoes the square this way.
  Check: $\sqrt{25} = 5$, since $5^2 = 25$.
- **Irrational numbers** (Class 9, Ganita Manjari, page 53).
  Here: $\sin 60° = \sqrt{3}/2$ has no exact fraction.
  Check: $\sqrt{2}$ squared is exactly 2, though $\sqrt{2}$ itself is not a fraction.
- **The difference of squares** (Class 9, Ganita Manjari, page 77).
  Here: a proof turns $(1 - \sin A)(1 + \sin A)$ into $1 - \sin^2 A$ this way.
  Check: $(5 - 2) \cdot (5 + 2) = 5^2 - 2^2$, both equal to 21.
- **Squaring a number** (Class 8, Ganita Prakash Part 1, page 21).
  Here: $\sin^2 A$ means $\sin A$ raised to the power 2, the notation used here.
  Check: 3 raised to the power 2 is 9.

If any of these felt new, read the page named before going on.

Nine plus sixteen is twenty-five



A square on each side of the triangle. The two squares on the legs together have exactly the area of the square on the hypotenuse.

Pythagoras is a statement about areas, and every identity in this chapter is that same statement divided through.

The three ratios of an acute angle

Before naming any ratio, we name the sides. The hypotenuse is the side opposite the right angle. It is the longest side, and its name never changes with A.

The other two sides are named relative to A itself. The opposite side faces A directly. The adjacent side also touches A, but it is not the hypotenuse.

We build three ratios from these three sides. $\sin A$ is opposite over hypotenuse. $\cos A$ is adjacent over hypotenuse. $\tan A$ is opposite over adjacent.

Pick a different acute angle in the same triangle, and the two names swap. Name the three sides correctly for the angle in question first: that is step one of every problem.

Your turn: in a right triangle, which side is adjacent to angle A? (Answer: the side that touches A, but is not the hypotenuse.)

BACK
Ratios of a right triangle · p. 251

TRY
Which side is “adjacent” to A?

Answer: The one touching A that is not the hypotenuse.

RULE
Opposite and adjacent are named relative to the ANGLE, never to the triangle as a whole.

CHECK YOURSELF

3. For an acute angle A in a right triangle, $\sin A$ is defined as

- **A.** the side adjacent to A over the hypotenuse
- **B.** the side opposite A over the hypotenuse
- **C.** the side opposite A over the side adjacent to A
- **D.** the hypotenuse over the side opposite A

4. Why does $\tan A = (\sin A)/(\cos A)$ **hold for every acute angle** A**?**

- **A.** because the hypotenuse cancels when $\sin A$ is divided by $\cos A$
- **B.** because $\tan A$ is defined that way by a separate rule unrelated to $\sin A$ and $\cos A$
- **C.** because the identity is only approximately true for small angles
- **D.** because $\cos A$ is always larger than $\sin A$

Reciprocal and quotient ratios

Three more ratios are just the first three turned upside down. $\text{cosec } A = 1/(\sin A)$. $\sec A = 1/(\cos A)$. $\cot A = 1/(\tan A)$. Each one is the reciprocal of a ratio we already named.

$\tan A$ also has a second life, as a quotient. $\tan A = (\sin A)/(\cos A)$. Its reciprocal follows the same pattern: $\cot A = (\cos A)/(\sin A)$.

The four newer names are bookkeeping on top of the first two, not four fresh ideas. If a ratio will not come to mind directly, rewrite it through $\sin A$ and $\cos A$ first. You will usually find this the fastest way in.

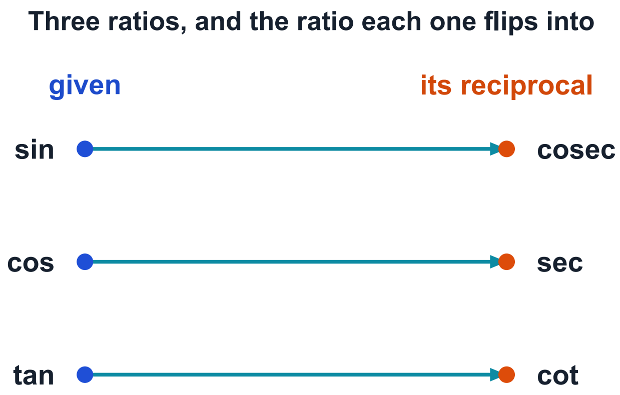
Your turn: what is $\sec A$ in terms of $\cos A$? (Answer: $\sec A = 1/(\cos A)$.)

TRY
What is $\cot A$ in terms of $\sin A$ and $\cos A$?

Answer: $(\cos A)/(\sin A)$.

RULE
If a ratio does not come to mind directly, rewrite it via $\sin A$ and $\cos A$ first.

KEY
$\text{cosec } A$, $\sec A$, $\cot A$ are just 1 over $\sin A$, $\cos A$, $\tan A$.



Any exercise asking for cosecant, secant or cotangent wants the reciprocal of a ratio you already found, not a fresh triangle measurement.

CHECK YOURSELF

5. In a right triangle, $\sin A = 3/5$ **and** $\cos A = 4/5$**. Using** $\cot A = (\cos A)/(\sin A)$**, what is** $\cot A$**?**

- **A.** $3/4$
- **B.** $4/3$
- **C.** $5/3$
- **D.** $5/4$

6. $\sec A$ **is defined as the reciprocal of**

- **A.** $\sin A$
- **B.** $\tan A$
- **C.** $\cot A$
- **D.** $\cos A$

2 more in this set online.

The ideas

Pythagoras' theorem

We use one Class 8 result throughout. For a right triangle with legs a and b and hypotenuse c, $c^2 = a^2 + b^2$. We do not prove it again here.

Identifying the hypotenuse correctly matters more than the formula itself. It is the side opposite the right angle, and it is always the longest of the three sides. Check this before substituting into the relation.

This same relation proves $\sin^2 A + \cos^2 A = 1$ soon, without being re-derived.

Your turn: a right triangle has legs 6 and 8. What is the hypotenuse? (Answer: $c = \sqrt{36 + 64} = 10$.)

CAUTION
We use this Class 8 result without re-proving it.

RULE
Identify the hypotenuse first: it is the side opposite the right angle, and the longest of the three.

KEY
The square of the hypotenuse equals the sum of the squares of the other two sides.

CHECK YOURSELF

7. In a right triangle with legs a **and** b **and hypotenuse** c**, Pythagoras' theorem states**

- **A.** $c = a^2 + b^2$
- **B.** $c^2 = a^2 - b^2$
- **C.** $a^2 = b^2 + c^2$
- **D.** $c^2 = a^2 + b^2$

8. This chapter proves $\sin^2 A + \cos^2 A = 1$ **using Pythagoras' theorem. What role does the theorem play in that proof?**

- **A.** it defines what $\sin A$ and $\cos A$ mean in the first place
- **B.** it proves that the triangle has a right angle
- **C.** it shows the triangle's angles add to 180°
- **D.** it supplies the side relation the proof relies on

1 more in this set online.

The ratios depend only on the angle

Every ratio so far rests on something not yet proved. $\sin A$, $\cos A$ and $\tan A$ depend only on angle A, never on the triangle's size. Let us prove it now, using similarity from Chapter 6.

THEOREM. Fix an acute angle A. Every right triangle carrying A gives the same $\sin A$, $\cos A$ and $\tan A$. The size of the triangle makes no difference.

We show it directly. Any two right triangles sharing angle A turn out similar to each other. That similarity then forces every ratio to match between them. The numbered proof below carries each step with its own reason.

Your turn: a 3-4-5 triangle and a 6-8-10 triangle both carry angle A. Should $\sin A$ come out the same for both? (Answer: yes, by the proof below.)

BACK
The three ratios of an acute angle · p. 254

TRY
A 3-4-5 triangle gives $\sin A = 3/5$. What does $\sin A$ come out to for its double, a 6-8-10 triangle?

Answer: $6/10 = 3/5$ again: the same triangle carrying angle A at a different size, exactly as the proof requires.

CAUTION
This proof leans on Chapter 6's AA similarity criterion, proved there, not here.

Given: We take two right triangles that both carry the same acute angle A. In the first, the side opposite A is p_1, the side adjacent to A is b_1, and the hypotenuse is h_1. In the second, the matching sides are p_2, b_2 and h_2.

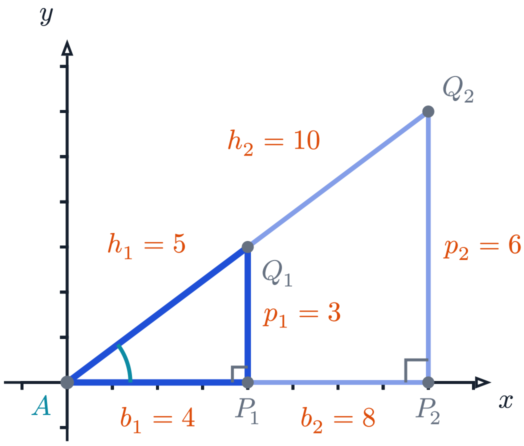
To Prove: For an acute angle A, the ratios $\sin A$, $\cos A$ and $\tan A$ are the same for every right triangle that carries A.

Proof:

	Statement	Reason
1.	let two right triangles both carry the same acute angle A	*In the first triangle, we call the side opposite A as p_1, the side adjacent to A as b_1, and the hypotenuse as h_1. We give the second triangle's matching sides the labels p_2, b_2 and h_2. The two triangles may be different sizes.*
2.	the two triangles are similar	*Both triangles have a right angle, and both have angle A. By AA similarity (Chapter 6), the triangles are similar.*
3.	$p_1/p_2 = b_1/b_2 = h_1/h_2$	*Similar triangles have proportional corresponding sides.*
4.	$p_1/h_1 = p_2/h_2$	*We rearrange $p_1/p_2 = h_1/h_2$ by cross-multiplying the proportion.*
5.	$\sin A$ measured in the first triangle equals $\sin A$ measured in the second	*The left side of the previous step is $\sin A$ in the first triangle. The right side is $\sin A$ in the second. This follows since $\sin A$ is opposite over hypotenuse, by definition.*
6.	$\cos A$ and $\tan A$ agree in both triangles too	*We apply the same rearrangement to $b_1/b_2 = h_1/h_2$ and to $p_1/p_2 = b_1/b_2$.*
7.	every right triangle carrying angle A gives the same three ratios	*The two triangles we compared were any two right triangles sharing A. So the argument holds for every such pair. Each ratio depends on the angle alone, never on the triangle's size.*

■

Same angle, two sizes, the same ratio



Because this holds for any two triangle sizes sharing an angle, a numeric example can pick the easiest size to compute with.

A SCHOLAR INDIA REMEMBERS

Draw a small right triangle with a $30°$ *angle. Then draw a big one. Will* $\sin 30°$ *come out the same? Count it. In the small one, the opposite side is* 1 *and the hypotenuse is* 2*. In the big one, they are* 3 *and* 6*. Both give* $1/2$*. The ratio depends on the angle, not on the size.*

Aryabhata · the hoopoe

CHECK YOURSELF

9. The proof that $\sin A$**,** $\cos A$ **and** $\tan A$ **depend only on the angle** A **rests on**

- **A.** the SSS congruence criterion, applied to two right triangles sharing angle A
- **B.** the AA similarity criterion, applied to two right triangles sharing angle A
- **C.** measuring the ratios in many different-sized triangles and averaging the results
- **D.** the fact that every right triangle has one angle equal to $90°$

10. In the proof, two right triangles share the same acute angle A **but have sides** p_1, b_1, h_1 **and** p_2, b_2, h_2**. Why does AA similarity apply to them?**

- **A.** both triangles have all three sides equal in length
- **B.** both triangles have the same area
- **C.** both triangles were drawn using the same ruler
- **D.** both share a right angle and angle A

3 more in this set online.

The ratios are bounded

The hypotenuse is the longest side of a right triangle. That one fact bounds every ratio we have defined so far.

For an acute angle A, $\sin A$ and $\cos A$ are each a shorter side over the hypotenuse. So both are positive, and never more than 1. Their reciprocals flip that bound: $\operatorname{cosec} A$ and $\sec A$ are always at least 1. $\tan A$ compares the two legs directly, with no hypotenuse involved, so it has no upper bound at all.

A claimed value like $\sin A = 4/3$ is impossible on sight. Check a ratio you compute against this bound: it catches an arithmetic slip early.

Your turn: is $\sec A = 2/3$ possible for an acute angle? (Answer: no. $\sec A$ can never be less than 1.)

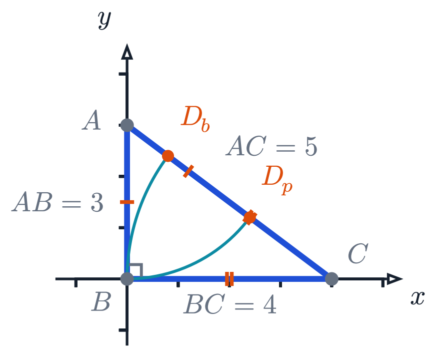
TRY
Is $\sec A = 2/3$ possible for an acute angle?

Answer: No. $\sec A$ can never be less than 1.

RULE
A claimed value like $\sin A = 4/3$ is impossible on sight: check against this bound first.

KEY
$\sin A$ and $\cos A$ never exceed 1; $\operatorname{cosec} A$ and $\sec A$ are never less than 1.

The hypotenuse always wins the race



This same argument holds on any right triangle: a leg divided by the hypotenuse can never reach or pass 1.

A SCHOLAR INDIA REMEMBERS

Put a bound on it before you work it out. In a right triangle the hypotenuse is the longest side. So opposite over hypotenuse is less than 1. That means $\sin A$ can never be 1.2. If you get 1.2, go back and look for the slip.

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CHECK YOURSELF

11. For an acute angle A, the value of $\sin A$

- **A.** can be greater than 1 for a large enough angle
- **B.** can be negative for some acute angles
- **C.** equals exactly 1 for every acute angle
- **D.** is always positive and never more than 1

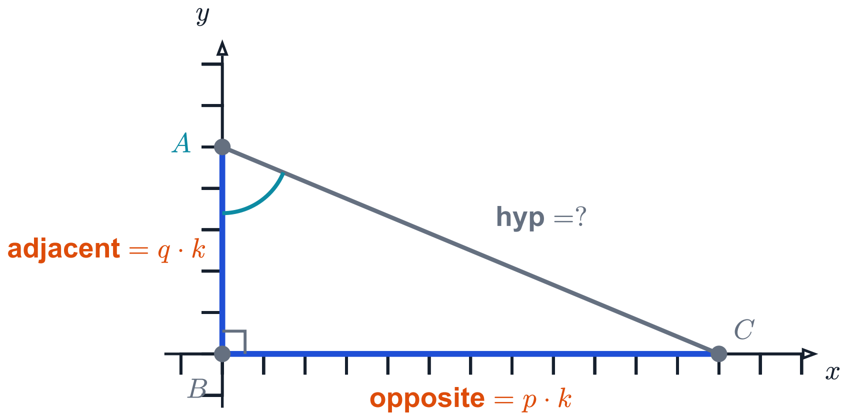
12. Why can $\sec A$ never be less than 1 for an acute angle A?

- **A.** because $\sec A$ is measured in different units from $\cos A$
- **B.** because $\sec A = 1/(\cos A)$ and $\cos A \leq 1$
- **C.** because the hypotenuse is always an integer
- **D.** because $\sec A$ is defined as the adjacent side over the opposite side

1 more in this set online.

From one ratio to every ratio

One ratio fixes two sides, up to a scale k



The hypotenuse is drawn thin and unlabeled because it is not yet known — it is Pythagoras' theorem's answer, not a number chosen up front.

This same two-sides-then-Pythagoras move is the one method behind any item that gives a single ratio and asks for the rest.

Given just one ratio of an acute angle A, we can recover every other ratio of that same angle. No triangle needs to be measured.

Write the given ratio as a fraction in lowest terms. Scale both the numerator and the denominator by the same positive number k. This fixes two of the triangle's three sides. Pythagoras' theorem then supplies the third side, and every ratio can be read directly off the three.

Scaling only one side breaks the right angle the whole recipe depends on. Given $\tan A = 5/12$, this fixes opposite $= 5k$ and adjacent $= 12k$. The hypotenuse is not yet found.

Your turn: given $\sin A = 5/13$, what two sides does this fix, before finding the third? (Answer: opposite $= 5k$, hypotenuse $= 13k$.)

AHEAD
Which ratio connects the two sides · p. 301

TRY
$\tan A = 5/12$. What are the two sides fixed by this, before finding the third?

Answer: Opposite $= 5k$, adjacent $= 12k$, for some positive k.

RULE
Always attach the same k to both sides named by the ratio, never to one side alone.

CHECK YOURSELF

13. Given that $\sin A = 2/3$ for an acute angle A, the standard recipe to find every other ratio starts by

- **A.** writing the opposite side as 2 and the adjacent side as 3
- **B.** measuring the actual triangle with a ruler to find the sides
- **C.** assuming the triangle is equilateral
- **D.** opposite $= 2k$, hypotenuse $= 3k$, for some $k > 0$

14. In the ratio-to-triangle recipe, after writing two sides as $2k$ and $3k$, why is Pythagoras' theorem used next?

A. to check whether the angle A is really acute

B. to convert the ratio into a decimal number

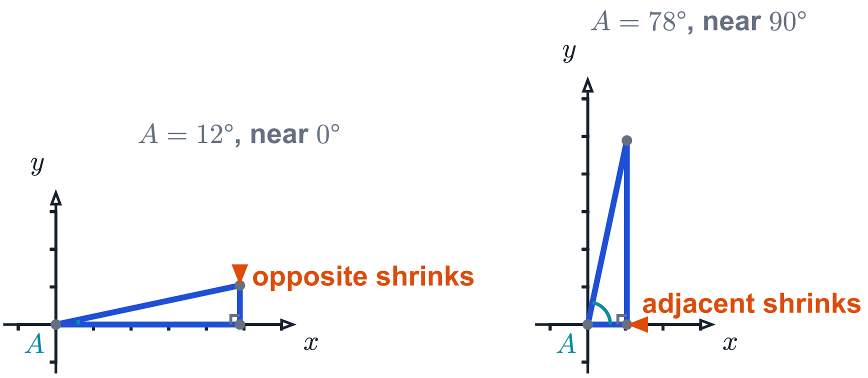
C. to find the triangle's third, still-missing side

D. because the two given sides might not actually form a right triangle

1 more in this set online.

Ratios at 0 and 90 degrees

Which side shrinks depends which way A moves



Neither triangle here can be drawn at exactly 0 degrees or 90 degrees, because no right triangle can carry that angle.

The ratios at $0°$ and $90°$ are not read off an actual triangle. No triangle can carry an angle of exactly $0°$ or $90°$ alongside its own right angle. Instead, we watch what happens as A approaches them.

As angle A shrinks toward $0°$, the side opposite A shrinks toward nothing. The adjacent side grows to nearly the full hypotenuse. This gives $\sin 0° = 0$ and $\cos 0° = 1$. Run the same reasoning as A grows toward $90°$. $\sin 90° = 1$ and $\cos 90° = 0$ follow the same way.

$\tan 90°$ is not defined, because its definition divides by $\cos 90° = 0$. Dividing by zero is never allowed.

From $\sin 0° = 0$ and $\cos 0° = 1$, $\tan 0° = 0$ follows directly. $\sec 90°$, $\operatorname{cosec} 0°$ and $\cot 0°$ fail for the same reason.

Your turn: why is $\operatorname{cosec} 0°$ undefined? (Answer: $\operatorname{cosec} 0° = 1/(\sin 0°) = 1/0$, and division by zero is not allowed.)

CAUTION
$\tan 90°$, $\sec 90°$, $\operatorname{cosec} 0°$ and $\cot 0°$ are all undefined.

RULE
A ratio is undefined only when its OWN division is by 0: check which ratio sits on the bottom.

KEY
$\sin$ grows from 0 to 1 as A runs $0°$ to $90°$; $\cos$ falls from 1 to 0.

CHECK YOURSELF

15. As angle A shrinks toward $0°$ in a right triangle, the opposite side shrinks to nothing while the adjacent side nearly equals the hypotenuse. This motivates

- **A.** $\sin 0° = 1$ and $\cos 0° = 0$
- **B.** $\sin 0° = 0$ and $\cos 0° = 0$
- **C.** $\sin 0° = 0$ and $\cos 0° = 1$
- **D.** $\tan 0° = 1$

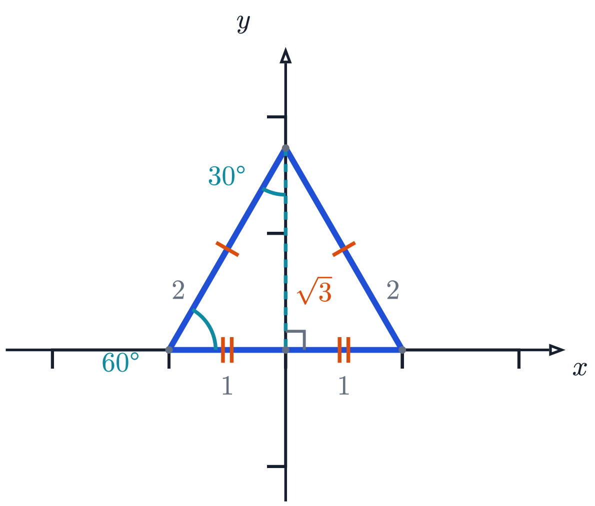
16. Why is $\tan 90°$ not defined, while $\tan 0°$ equals 0?

- **A.** because no right triangle can ever contain a $90°$ angle
- **B.** because $\tan A$ is only defined for angles below $45°$
- **C.** because dividing by $\cos 90° = 0$ is not allowed
- **D.** because $\sin 90°$ is also not defined

1 more in this set online.

The standard-angle table

Bisect an equilateral triangle, get $30°$ and $60°$



Every 30 degree and 60 degree value in the standard-angle table comes from this one triangle, so a forgotten entry can be rebuilt.

Five angles carry exact values worth memorising: $0°$, $30°$, $45°$, $60°$ and $90°$.

Sine runs 0, then $1/2$, then $1/\sqrt{2}$, then $\sqrt{3}/2$, then 1. Cosine runs the same five numbers in reverse: 1, then $\sqrt{3}/2$, then $1/\sqrt{2}$, then $1/2$, then 0. $\tan A$ is $\sin A$ divided by $\cos A$ at each angle: 0, then $1/\sqrt{3}$, then 1, then $\sqrt{3}$. $\tan A$ has no value at $90°$, since $\cos A$ is 0 there. Every reciprocal ratio follows by inverting the matching entry.

The two rows run through the same five numbers in opposite order. We do not prove why here, though it is worth noticing. Working $\tan A$ out from $\sin A$ and $\cos A$ is usually faster than memorising a third list. A right triangle with two equal legs gives $45°$. Half an equilateral triangle gives $30°$ and $60°$ instead.

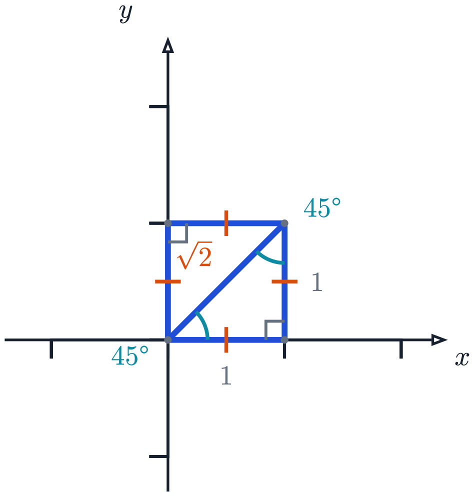
Your turn: what is $\cos 60°$? (Answer: $1/2$, reading the cosine row at $60°$.)

BACK
The three ratios of an acute angle · p. 254

AHEAD
Worked: a tower's height from one triangle · p. 305

RULE
tan is sin over cos at each angle: derive it rather than memorise a third list.

A square's diagonal gives $45°$ twice



The 45 degree column of the standard-angle table comes from exactly this triangle, with no other construction needed.

A SCHOLAR INDIA REMEMBERS

Aryabhata's mathematics rested on geometric reasoning. A result was accepted because it could be worked out, not because someone important said so. You can work out this table the same way. Take an equilateral triangle with sides 2. Cut it in half. The half has angles $30°$, $60°$ and $90°$. The side opposite $30°$ is 1, and the hypotenuse is 2. So $\sin 30° = 1/2$.

Aryabhata · the hoopoe

CHECK YOURSELF

17. At $A = 45°$, the values of $\sin A$ and $\cos A$ are

- **A.** both equal to $1/\sqrt{2}$
- **B.** $\sin 45° = 1/2$ and $\cos 45° = \sqrt{3}/2$
- **C.** $\sin 45° = \sqrt{3}/2$ and $\cos 45° = 1/2$
- **D.** $\sin 45° = 1$ and $\cos 45° = 0$

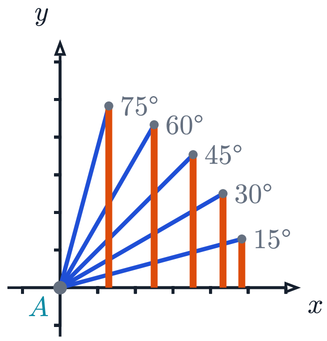
18. The standard-angle table lists tan as not defined at $90°$ but gives an exact value at every other listed angle. Why?

- **A.** because no triangle can have an angle of exactly $90°$
- **B.** because $\sin 90°$ is undefined, and tan needs sin
- **C.** because $\cos 90° = 0$ makes the division impossible
- **D.** because the table only lists five angles by convention, with no deeper reason

1 more in this set online.

One ratio value, one angle

Five angles, five heights, none repeated



Because no two of these five heights repeat, one ratio value always points back to exactly one angle, never two.

Acute angle A runs from $0°$ to $90°$. As it does, $\sin A$ climbs from 0 to 1. $\cos A$ falls from 1 to 0 over the same range. Neither ratio ever doubles back.

Two right triangles giving the same value of $\sin A$ must carry the same angle A. The same holds for $\cos A$.

No two different acute angles ever share either ratio. This turns a numerical match into a statement about angles. If we can show two acute angles share one ratio value, that already shows the angles themselves are equal.

Your turn: two acute angles have the same $\cos$ value. Are the angles equal? (Answer: yes, since $\cos$ takes each value at exactly one acute angle.)

BACK
The ratios depend only on the angle · p. 256

TRY
Two acute angles have the same $\cos$ value. Are the angles equal?

Answer: Yes. $\cos$ takes each value at exactly one acute angle.

RULE
To show two acute angles are equal, it is enough to show they share one ratio value.

CHECK YOURSELF

19. As acute angle A runs from $0°$ to $90°$, $\sin A$

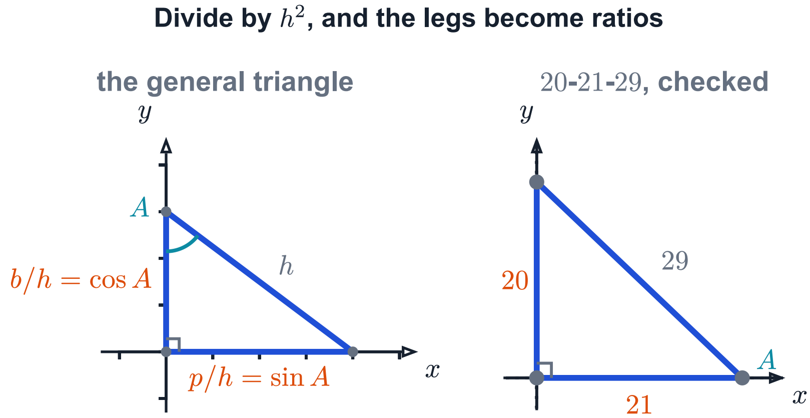
- **A.** falls steadily from 1 to 0
- **B.** rises steadily from 0 to 1
- **C.** rises, then falls back to 0
- **D.** stays constant at 1/2

20. Two right triangles give the same value of $\cos A$. What can be concluded about the two angles?

- **A.** the two angles could be different, since many angles can share the same $\cos A$
- **B.** the two triangles must be congruent
- **C.** nothing can be concluded without also comparing $\sin A$
- **D.** the two angles must be equal, since $\cos A$ never repeats

1 more in this set online.

The Pythagorean trigonometric identity



In the numeric panel, A is the angle opposite the leg of 20, which is why sin A equals 20/29, not 21/29.

One identity ties $\sin A$ and $\cos A$ together, for every acute angle. It follows directly from Pythagoras' theorem.

THEOREM. For any acute angle A of a right triangle, $\sin^2 A + \cos^2 A = 1$.

We reproduce this one proof in full. Every other identity we meet is built from it. The numbered proof below divides Pythagoras' theorem by the hypotenuse squared, then simplifies. Read it slowly the first time through.

Your turn: for a 5-12-13 triangle, does $\sin^2 A + \cos^2 A$ come out to 1? (Answer: yes. $(5/13)^2 + (12/13)^2 = 25/169 + 144/169 = 1$.)

TRY
For a 3-4-5 triangle, $\sin A = 3/5$ and $\cos A = 4/5$. What does $\sin^2 A + \cos^2 A$ come out to?

Answer: $(3/5)^2 + (4/5)^2 = 9/25 + 16/25 = 1$: exactly as the identity requires.

CAUTION
Divide by h^2, not by p^2 or b^2: those give the OTHER two identities, not this one.

RULE
This is the ONE proof to reproduce in full: every other identity is built from it.

Given: We take a right triangle with acute angle A. Call the side opposite A as p, the side adjacent to A as b, and the hypotenuse as h.

To Prove: For any acute angle A of a right triangle, $\sin^2 A + \cos^2 A = 1$.

Proof:

	Statement	Reason
1.	let a right triangle have acute angle A, opposite side p, adjacent side b, hypotenuse h	*We name the three sides relative to A, exactly as in the ratio definitions.*
2.	$p^2 + b^2 = h^2$	*This is Pythagoras' theorem, stated earlier.*
3.	$p^2/h^2 + b^2/h^2 = h^2/h^2$	*We divide both sides of the equation by the same nonzero number, h^2. This preserves the equation.*
4.	$(p/h)^2 + (b/h)^2 = 1$	*This follows from the laws of exponents, since $h^2/h^2 = 1$.*
5.	$\sin^2 A + \cos^2 A = 1$	*By definition, $p/h = \sin A$ and $b/h = \cos A$.*

6.	the identity holds for every acute angle	*A was any acute angle of any right triangle. Nothing in the argument depended on a particular triangle.*

■

CHECK YOURSELF

21. The identity $\sin^2 A + \cos^2 A = 1$ holds for

- **A.** only the angle $A = 45°$
- **B.** only right triangles with integer side lengths
- **C.** every acute angle A of a right triangle
- **D.** only when the triangle is isosceles

22. The proof of $\sin^2 A + \cos^2 A = 1$ starts from $p^2 + b^2 = h^2$ and divides every term by h^2. What does this division achieve?

- **A.** it finds the numeric value of the hypotenuse h
- **B.** it proves that p and b are equal
- **C.** it turns the sides into $\sin A$ and $\cos A$
- **D.** it changes the right angle into an acute angle

2 more in this set online.

Two more identities from one

The same Pythagoras relation, $p^2 + b^2 = h^2$, gives two more identities by dividing by a different squared side.

Divide by b^2, the adjacent side's square, and we reach $1 + \tan^2 A = \sec^2 A$. Divide the same relation by p^2, the opposite side's square. The result is $1 + \cot^2 A = \operatorname{cosec}^2 A$ instead.

$1 + \tan^2 A = \sec^2 A$ holds for every acute A. $1 + \cot^2 A = \operatorname{cosec}^2 A$ needs $\cot A$ and $\operatorname{cosec} A$ to be defined. This rules out $A = 0°$.

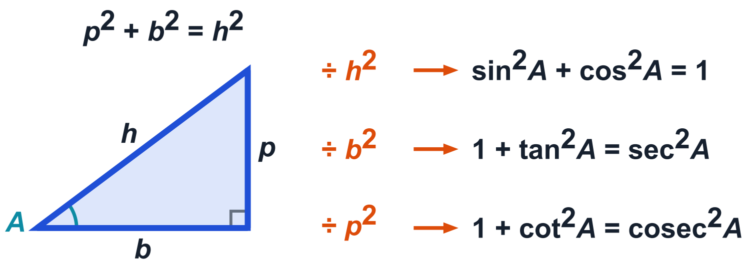
Your turn: dividing $p^2 + b^2 = h^2$ by p^2 gives which identity? (Answer: $1 + \cot^2 A = \operatorname{cosec}^2 A$.)

KEY
Divide $p^2 + b^2 = h^2$ by b^2 for tan/sec; by p^2 for cot/cosec.

CAUTION
$1 + \cot^2 A = \operatorname{cosec}^2 A$ fails at $A = 0°$: cot and cosec are undefined there.

RULE
All three identities divide the same Pythagoras relation by a different squared side.

Three identities from one relation



one relation, three squared sides to divide by

Pythagoras' theorem is one relation between three sides. Divide it by the square of the hypotenuse, the adjacent side or the opposite side and you get the three identities, one each. Nothing new is assumed.

All three identities are one theorem divided three ways: Pythagoras' relation shared out by the square of whichever side you choose.

CHECK YOURSELF

23. Dividing $p^2 + b^2 = h^2$ by b^2, the square of the adjacent side, gives

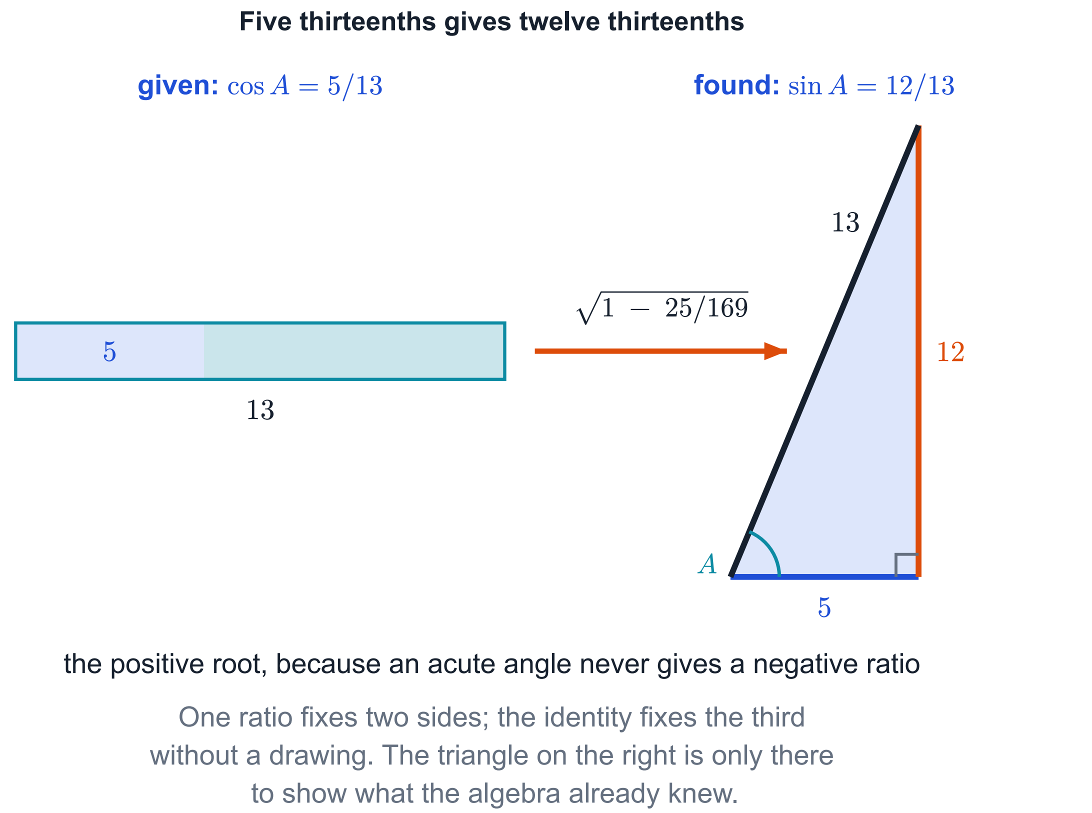
- **A.** $1 + \cot^2 A = \text{cosec}^2 A$
- **B.** $\sin^2 A + \cos^2 A = 1$
- **C.** $1 - \tan^2 A = \sec^2 A$
- **D.** $1 + \tan^2 A = \sec^2 A$

24. Both $1 + \tan^2 A = \sec^2 A$ and $1 + \cot^2 A = \text{cosec}^2 A$ come from the same source. What is it?

- **A.** two separate, unrelated proofs, one for each identity
- **B.** measurement of many right triangles
- **C.** the standard-angle table's exact values
- **D.** Pythagoras' theorem, divided differently each time

1 more in this set online.

Any one ratio gives the rest, algebraically



Given cos A equals 5/13, the identity forces sin A equals 12/13, taking the positive root for an acute angle.

Once we know one ratio of an acute angle, the Pythagorean identity gives every other ratio algebraically. No triangle needs to be drawn at all.

From $\sin A$ alone, $\cos A = \sqrt{1 - \sin^2 A}$. Take the positive root: you already know every ratio of an acute angle is positive. Every other ratio then follows from these two, using the reciprocal and quotient relations already met.

An acute angle never gives a negative ratio. The negative root is never the answer here.

Your turn: given $\cos A = 5/13$, what is $\sin A$? (Answer: $\sin A = \sqrt{1 - 25/169} = 12/13$.)

BACK
The Pythagorean trigonometric identity · p. 265

TRY
Given $\cos A$, how is $\sin A$ found algebraically?

Answer: $\sin A = \sqrt{1 - \cos^2 A}$, the positive root.

RULE
Reduce to $\sin A$ and $\cos A$ first, then build any other ratio from those two.

CHECK YOURSELF

25. Given only $\sin A$ for an acute angle A, every other ratio can be found because

- **A.** $\cos A = \sqrt{1 - \sin^2 A}$ gives every other ratio
- **B.** a triangle must first be drawn and measured with a ruler
- **C.** $\tan A$ must be looked up separately in the standard-angle table
- **D.** $\cos A$ and $\sin A$ are actually independent and unrelated

26. Why is only the positive square root taken when writing $\cos A = \sqrt{1 - \sin^2 A}$ for an acute angle A?

A. because square roots are always defined as positive by convention, regardless of context

B. because $1 - \sin^2 A$ is always a perfect square

C. because A is acute, so $\cos A$ must be positive, and the negative root would be wrong

D. because $\cos A$ is always larger than $\sin A$

1 more in this set online.

Common mistakes

Assuming the identity you are proving

There is a tempting shortcut when proving an identity. Manipulate both sides at once, as though they were already equal. Cross-multiply. Simplify. Stop once the two sides match.

That method proves nothing. If we treat an unproved identity as already true, we assume the very thing the proof was meant to show. The match at the end was fixed from the first step, not earned.

A genuine proof works from one side alone, using only identities already established, until that side becomes the other.

The statement being proved never appears anywhere in the middle of the working. The worked example proving $\sec A(1 - \sin A)(\sec A + \tan A) = 1$ follows exactly this rule.

Your turn: a working assumes $\sec A(1 - \sin A)(\sec A + \tan A) = 1$ from the start. Then it simplifies both sides until they match. Is this a valid proof? (Answer: no. It assumes the very thing it set out to prove.)

CAUTION
Cross-multiplying an unproved identity treats it as already true: exactly the trap.

KEY
Reduce ONE side only; never touch both sides of the identity you have not yet proved.

FIRST TRY
~~To prove $1 + \tan^2 A = \sec^2 A$ for $A = 30°$, I assumed the two sides equal, then simplified both sides together until they matched. $\sin^2 A + \cos^2 A = 1$ came out at the end, so I called the identity proved.~~

SECOND LOOK
That assumes the very thing it sets out to prove. Working from the left side alone: $1 + \tan^2 A = 1 + (\sin^2 A)/(\cos^2 A) = (\cos^2 A + \sin^2 A)/(\cos^2 A) = 1/(\cos^2 A) = \sec^2 A$. For $A = 30°$, this gives $1 + 1/3 = 4/3$, exactly $\sec^2 30°$.

Work from one side alone. The identity being proved must never appear until the very last line.

DOES WORKING ON BOTH SIDES AT ONCE EVER PROVE AN IDENTITY?

WEAK

We want to prove $1 + \tan^2 A = \sec^2 A$. Start by assuming both sides are equal. Write $1 + (\sin A)^2/(\cos A)^2 = 1/(\cos A)^2$. Subtract 1 from both sides: $(\sin A)^2/(\cos A)^2 = 1/(\cos A)^2 - 1$. Multiply both sides by $\cos^2 A$: $\sin^2 A = 1 - \cos^2 A$. This is just $\sin^2 A + \cos^2 A = 1$, already known. The two sides match, so the identity is "proved."

The catch: line two already treated $1 + \tan^2 A = \sec^2 A$ as true, before it was shown. Everything after that just restates a fact already known. Nothing here shows the identity itself is true.

STRONG

We want to prove $1 + \tan^2 A = \sec^2 A$, working from the left side alone. $1 + \tan^2 A = 1 + (\sin^2 A)/(\cos^2 A)$, writing $\tan A$ in terms of $\sin A$ and $\cos A$. Combine over one denominator: $= (\cos^2 A + \sin^2 A)/(\cos^2 A)$. Use $\sin^2 A + \cos^2 A = 1$ once, in the numerator: $= 1/(\cos^2 A)$. This is $\sec^2 A$, the right side.

The identity being proved never appeared until the very last line, where the left side finished equal to the right side.

A SCHOLAR INDIA REMEMBERS

You wrote the identity at the top, then showed it was true. Did you earn the answer, or did you assume it? Start from one side only. Change it one step at a time. Stop when it has become the other side. Now the identity is earned.

Aryabhata · the hoopoe

CHECK YOURSELF

27. A student proves $\sec A - \tan A \cdot \sin A = \cos A$ by writing: "Cross-multiplying both sides, $\sec A \cdot \cos A - \tan A \cdot \sin A \cdot \cos A = \cos A \cdot \cos A$, which simplifies to $1 - \sin^2 A = \cos^2 A$, true by the Pythagorean identity, so the original statement is proved." What is wrong with this method?

- **A.** nothing is wrong — manipulating both sides of an identity at once is a standard, valid way to confirm it
- **B.** the final simplified equation, $1 - \sin^2 A = \cos^2 A$, is actually false
- **C.** $\sec A$ was defined incorrectly in the working
- **D.** it manipulates both sides of the identity at once, as if they were already known to be equal

28. Why does "manipulate both sides at once until they match" fail as a proof method for an identity, even when every algebraic step is individually correct?

- **A.** because algebra performed on both sides of an equation is never actually valid
- **B.** because it only works for identities involving $\sin$ and $\cos$, not other ratios
- **C.** because it assumes the very equality it sets out to prove
- **D.** because it takes too many steps compared to working from one side alone

2 more in this set online.

Sine notation is not multiplication

$\sin A$ looks, at a glance, like two things side by side. It is not "sin" multiplied by A. Read that way, a step like $\sin A \cdot \cos A$ might tempt us to cancel the A's. That attempt means nothing.

$\sin A$, $\cos A$, $\tan A$, $\operatorname{cosec} A$, $\sec A$ and $\cot A$ are each one inseparable symbol. Each is read as "the sine of angle A," and so on. "Sin" written alone, with no angle attached, names nothing at all. It can never be split off, multiplied, cancelled, or factored on its own.

Your turn: in $(\sin A)/(\cos A)$, is it valid to cancel the two A's? (Answer: no. $\sin A$ and $\cos A$ are each one symbol; there is no lone A to cancel.)

IN SINE OF A DIVIDED BY SINE OF B, CAN THE SINE BE CANCELLED, LEAVING A OVER B?

WEAK
A learner sees $(\sin A)/(\sin B)$ and reads it as "sin" times A, divided by "sin" times B. Cancelling the "sin" from top and bottom, they write $(\sin A)/(\sin B) = A/B$. For $A = 30°$ and $B = 60°$, this gives $A/B = 1/2$.

Check it against real values: $\sin 30° = 1/2$ and $\sin 60° = \sqrt{3}/2$, so $(\sin A)/(\sin B) = (1/2)/(\sqrt{3}/2) = 1/\sqrt{3}$, not $1/2$. The cancelled "sin" was never a separate factor to cancel: there is no lone "sin" standing anywhere in $(\sin A)/(\sin B)$.

STRONG
$\sin A$ and $\sin B$ are each one number, not a product. $(\sin A)/(\sin B)$ is one number divided by another, and nothing in it cancels. For $A = 30°$ and $B = 60°$, $(\sin A)/(\sin B) = (1/2)/(\sqrt{3}/2) = 1/\sqrt{3}$.

That is the actual value, found by computing each sine first, then dividing. No letter or symbol was cancelled at any step.

BACK
The three ratios of an acute angle · p. 254

CAUTION
"sin", "cos", "tan" (and their reciprocals) never stand alone: they always carry an angle.

KEY
$\sin A$ names ONE ratio; it is not two things multiplied together.

WATCH OUT

The trap: In $\sin A/A$, the student cancels the A on top and bottom, leaving a bare sin.

The correction: $\sin A$ is a single symbol, not "sin" times A. There is no lone "sin" anywhere in it to cancel, multiply, or factor out.

CHECK YOURSELF

29. A student simplifies $\sin A/A$ by "cancelling" the A, writing the result as sin. What is wrong with this step?

- **A.** the step is valid, since $\sin A$ really does mean "sin" multiplied by A
- **B.** the student used the wrong ratio; the step would be valid for $\tan A/A$ instead
- **C.** the student forgot to convert A from degrees to radians before cancelling
- **D.** $\sin A$ is one inseparable symbol, not "sin" times A

30. Which of these is a correct way to read the symbol $\sin A$?

- **A.** "sin" multiplied by A
- **B.** "the sine of angle A", one inseparable symbol
- **C.** the letter "sin" raised to the power A
- **D.** the value of A measured in a unit called "sin"

2 more in this set online.

Worked examples

WORKED EXAMPLE · **All six ratios from tan A = 4/3**

1.	$\tan A = 4/3$, so opposite $= 4k$ and adjacent $= 3k$ for some positive k	*$\tan A$ is opposite over adjacent. The given ratio fixes these two sides, up to a scale factor k.*
2.	hypotenuse $= \sqrt{(4k)^2 + (3k)^2} = \sqrt{25k^2} = 5k$	*Pythagoras' theorem supplies the third side.*

WORKED EXAMPLE · **All six ratios from tan A = 4/3**

3. $\sin A = 4/5$, $\cos A = 3/5$, $\tan A = 4/3$ — *We read the three ratios directly off the three sides. The k cancels in every ratio.*

4. $\operatorname{cosec} A = 5/4$, $\sec A = 5/3$, $\cot A = 3/4$ — *Each is the reciprocal of the matching ratio above.*

5. $(4/5)^2 + (3/5)^2 = 16/25 + 9/25 = 1$ — CHECK ***Check by squaring** $\sin A$ **and** $\cos A$ **and adding them.** They must add to 1, and here they do.*

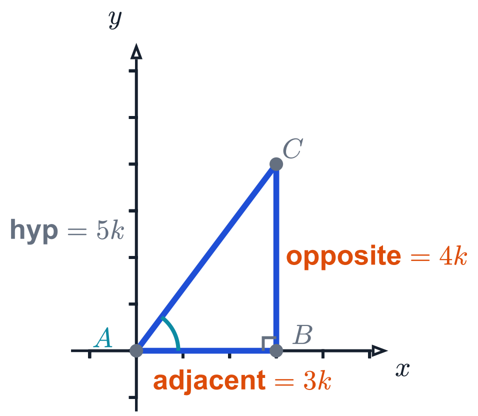
BACK
From one ratio to every ratio · p. 260

TRY
If $\tan A = 5/12$ for an acute angle, what is $\sin A$?

Answer: $\sin A = 5/13$, from the 5, 12, 13 triangle.

RULE
Scale the ratio into two sides first, find the third with Pythagoras' theorem, then read every other ratio off the three.

$\tan A = 4/3$ **becomes a** $4k$**-**$3k$**-**$5k$ **triangle**



Keeping the scale factor attached to every side is why these three ratios stay identical no matter how large the triangle is built.

GIVEN THE TANGENT OF AN ANGLE, FIND THE OTHER FIVE RATIOS.

1. **Scale into two sides** For $\tan A = p/q$, set opposite $= pk$ and adjacent $= qk$, for some positive k. Here $\tan A = 4/3$ gives opposite $= 4k$, adjacent $= 3k$.
2. **Find the hypotenuse** Use Pythagoras' theorem, hypotenuse $= \sqrt{(pk)^2 + (qk)^2}$. Here that is $\sqrt{(4k)^2 + (3k)^2} = 5k$.
3. **Read off sine and cosine** $\sin A$ is opposite over hypotenuse; $\cos A$ is adjacent over hypotenuse. Here $\sin A = 4/5$ and $\cos A = 3/5$.
4. **Take the three reciprocals** $\operatorname{cosec} A = 1/(\sin A)$, $\sec A = 1/(\cos A)$, $\cot A = 1/(\tan A)$. Here that gives $5/4$, $5/3$ and $3/4$.
5. **Check against the identity** Square $\sin A$ and $\cos A$ and add them. They must equal 1. Here $(4/5)^2 + (3/5)^2 = 1$.

CHECK YOURSELF

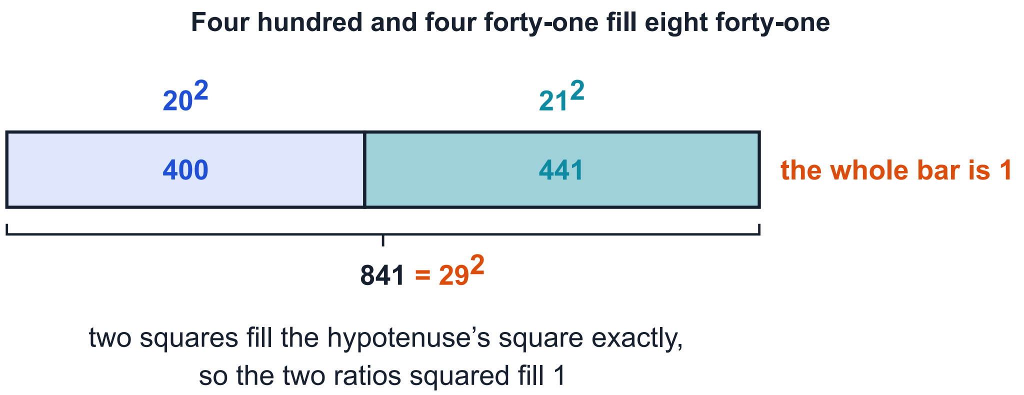
31. Given $\tan A = 4/3$ for an acute angle A, the standard recipe writes the opposite and adjacent sides as

- **A.** 4 and 3, exactly, with no scaling factor
- **B.** $4k$ and $3k$, for some positive k
- **C.** $3k$ and $4k$, with adjacent listed first
- **D.** $4k$ and $3k$, naming the hypotenuse and one leg

32. Given opposite $= 4k$ and adjacent $= 3k$, the hypotenuse is found as $\sqrt{(4k)^2 + (3k)^2} = 5k$. Why is $5k$ specifically, and not another multiple of k, the correct hypotenuse?

- **A.** because every right triangle with legs in ratio 4 to 3 must have hypotenuse 5 units long
- **B.** because $(4k)^2 + (3k)^2 = 25k^2$, and its square root is exactly $5k$
- **C.** because $4 + 3 = 7$, and 5 is roughly half of 7
- **D.** because 5 is the next prime number after 4 and 3

1 more in this set online.



$\sin^2 A$ and $\cos^2 A$ are the two parts of one whole. On the 20-21-29 triangle the parts are 400 and 441 out of 841, and they leave nothing over, which is what the identity promises for every angle.

Checking the identity on whole numbers checks that two parts fill a whole: 400 and 441 make 841, so the squared ratios make exactly 1.

WORKED EXAMPLE · **Checking the identity on a 20-21-29 triangle**

1.	hypotenuse 29, one leg 21, so the other leg is $\sqrt{29^2 - 21^2} = \sqrt{841 - 441} = \sqrt{400} = 20$	*Pythagoras' theorem, rearranged to find the missing leg.*
2.	for the angle A opposite the leg 20: $\sin A = 20/29$, $\cos A = 21/29$	*We read the ratios off the three sides of this triangle.*
3.	$(20/29)^2 + (21/29)^2 = 400/841 + 441/841 = 841/841 = 1$	CHECK ***This confirms*** $\sin^2 A + \cos^2 A = 1$ ***numerically****, on a triangle where every side is a whole number.*

BACK
The Pythagorean trigonometric identity · p. 265

TRY
Hypotenuse 17, one leg 15: find the other leg, then check $\sin^2 A + \cos^2 A = 1$ for the angle A opposite it.

Answer: Other leg $\sqrt{17^2 - 15^2} = 8$; $\sin A = 8/17$, $\cos A = 15/17$, and $(64 + 225)/289 = 1$.

CAUTION
One numeric triangle confirms the identity here: it does not replace the general proof for every angle A.

CHECK THE PYTHAGOREAN IDENTITY ON A RIGHT TRIANGLE WITH WHOLE-NUMBER SIDES.

1. **Confirm the triangle is right-angled** The identity only holds for a right triangle. Check a right angle is marked before using Pythagoras' theorem.
2. **Find the side not given** Use $c^2 = a^2 + b^2$ to find the missing side. For hypotenuse 29 and leg 21, the missing leg is $\sqrt{29^2 - 21^2} = 20$.
3. **Read sine and cosine off the three sides** Pick the angle A at one acute vertex. $\sin A$ is opposite over hypotenuse; $\cos A$ is adjacent over hypotenuse. Here $\sin A = 20/29$, $\cos A = 21/29$.
4. **Square and add** Check $\sin^2 A + \cos^2 A = 1$ using the fractions from the last step. Here $(20/29)^2 + (21/29)^2 = 1$.

CHECK YOURSELF

33. A right triangle has hypotenuse 29 and one leg 21. Using Pythagoras' theorem, the other leg is

- **A.** $\sqrt{50}$
- **B.** 8
- **C.** 20
- **D.** 50

34. For the triangle with sides 20, 21 and hypotenuse 29, $\sin A = 20/29$ and $\cos A = 21/29$ give $(20/29)^2 + (21/29)^2 = 1$. What does this numeric check confirm?

- **A.** it confirms the identity in this one triangle
- **B.** that the identity is proved true for every acute angle by this one example
- **C.** that 20, 21 and 29 are the only side lengths for which the identity holds
- **D.** that $\sin A$ and $\cos A$ must always add to 1, not just their squares

1 more in this set online.

WORKED EXAMPLE · **Reading two sides off the standard-angle table**

1.	in ABC, right-angled at B, angle $C = 30°$ and AB $= 5$ cm	*This is the given information. The right angle and the known angle are both named.*
2.	AB is opposite C and BC is adjacent to C	*We name the sides relative to the angle we are given, C, not to the right angle at B.*
3.	$\tan 30° = \text{AB}/\text{BC}$, so $\text{BC} = \text{AB}/(\tan 30°) = 5\sqrt{3}$ cm	$\tan 30° = 1/\sqrt{3}$, *from the standard-angle table.*

WORKED EXAMPLE · **Reading two sides off the standard-angle table**

4. $\sin 30° = AB/AC$, so $AC = AB/(\sin 30°) = 10$ cm

$\sin 30° = 1/2$, from the same table.

5. $5^2 + \left(5\sqrt{3}\right)^2 = 25 + 75 = 100 = 10^2$

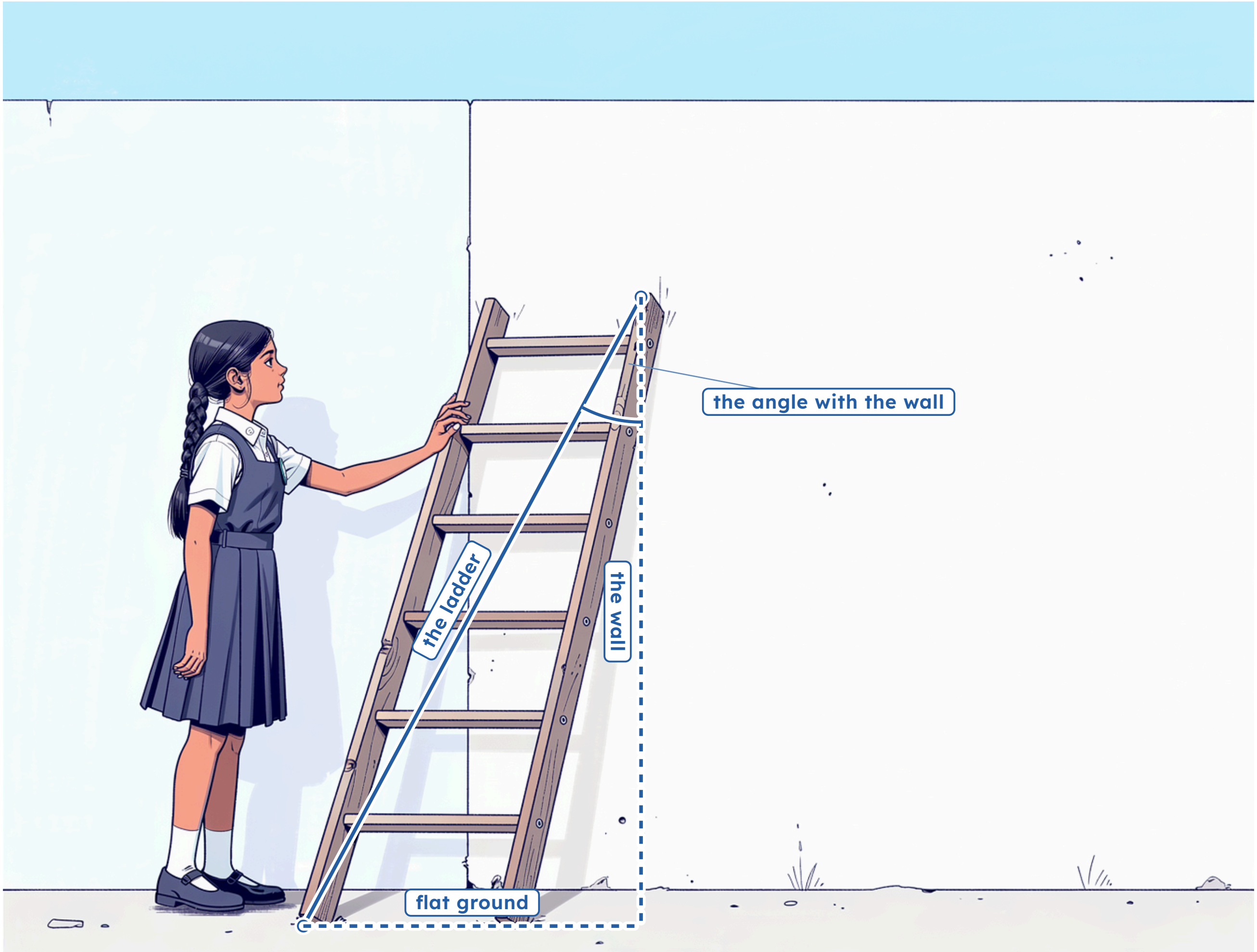
CHECK ***Check both sides found against Pythagoras' theorem.*** *They should give the hypotenuse back, and here they do.*

BACK
The standard-angle table · p. 262

TRY
If angle $C = 45°$ and $AB = 5$ cm, what is BC?

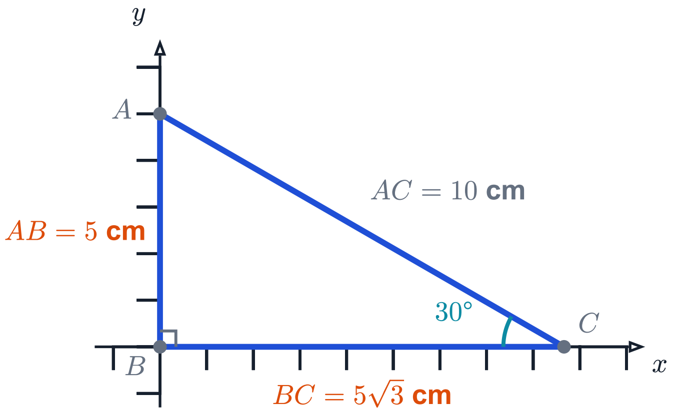
Answer: $BC = 5$ cm, since $\tan 45° = 1$.

RULE
Name the side opposite the known angle first, then choose which ratio pairs it with the unknown side.



The ladder is the hypotenuse. The wall and the flat ground are the triangle's other two sides.

One side, one standard angle, the whole triangle



The same two ratios used here to solve for the missing sides are the ones most missing-side problems reach for first.

FIND TWO UNKNOWN SIDES USING THE STANDARD-ANGLE TABLE.

1. **Name the sides relative to the given angle** Call the side opposite the given angle "opposite," and the side touching it that is not the hypotenuse "adjacent." For angle $C = 30°$ with $\mathrm{AB} = 5$ cm opposite C, BC is adjacent to C.
2. **Pick a ratio linking a known side to an unknown one** Read the ratio for the given angle off the standard-angle table. Here $\tan 30° = \mathrm{AB}/\mathrm{BC}$ gives $\mathrm{BC} = 5\sqrt{3}$ cm.
3. **Pick a second ratio for the last side** Use a different ratio at the same angle. Here $\sin 30° = \mathrm{AB}/\mathrm{AC}$ gives $\mathrm{AC} = 10$ cm.
4. **Check with Pythagoras' theorem** The three sides found must satisfy $a^2 + b^2 = c^2$. Here $5^2 + \left(5\sqrt{3}\right)^2 = 10^2$.

CHECK YOURSELF

35. In triangle ABC**, right-angled at** B**, with angle** $C = 30°$ **and** $\mathrm{AB} = 5$ **cm, the side** AB **is**

- **A.** adjacent to angle C
- **B.** the hypotenuse of the triangle
- **C.** opposite to angle C
- **D.** equal in length to BC

36. In the same triangle, $\tan 30° = \mathrm{AB}/\mathrm{BC}$ **is used to find** BC**. Why is** tan**, rather than** sin **or** cos**, the right ratio to use here?**

- **A.** because tan always gives a larger, more accurate answer than sin or cos
- **B.** because the angle given is exactly 30°, and tan only works for that angle
- **C.** tan links the known opposite side to the wanted adjacent side
- **D.** because sin and cos cannot be used when the hypotenuse is unknown

1 more in this set online.

WORKED EXAMPLE · **Checking 2 sin A cos A = 1 when tan A = 1**

1. $\tan A = 1$, so the two legs of the right triangle are equal

$\tan A$ is opposite over adjacent. That ratio is 1 only when the two legs match.

WORKED EXAMPLE · Checking 2 sin A cos A = 1 when tan A = 1

2. let each leg be k, so the hypotenuse is $\sqrt{k^2+k^2} = k\sqrt{2}$ — *Pythagoras' theorem, with both legs equal to k.*

3. $\sin A = k/(k\sqrt{2}) = 1/\sqrt{2}$, $\cos A = 1/\sqrt{2}$ — *We read both ratios off the three sides. The k cancels.*

4. $2 \cdot \sin A \cdot \cos A = 2 \cdot \left(1/\sqrt{2}\right) \cdot \left(1/\sqrt{2}\right) = 2/2 = 1$ — CHECK ***Check by substituting both ratios back into $2 \sin A \cos A$.** It simplifies to exactly 1.*

CAUTION
This confirms $2 \cdot \sin A \cdot \cos A = 1$ only at $A = 45°$: it is not a general identity for every acute angle.

RULE
Build the triangle from $\tan A = 1$ first, using equal legs, before computing $\sin A$ and $\cos A$.

CHECK YOURSELF

37. If $\tan A = 1$ for an acute angle A in a right triangle, the two legs are

- **A.** in the ratio 2 to 1
- **B.** equal in length
- **C.** always exactly 1 unit long
- **D.** impossible, since no two sides of a right triangle can be equal

38. With both legs equal to k, the hypotenuse works out to $k\sqrt{2}$. Why does $\sin A$ then equal $1/\sqrt{2}$, not simply 1?

- **A.** because $\sin A$ is always smaller than $\tan A$, whatever the triangle
- **B.** $\sin A$ is opposite over hypotenuse, $k/(k\sqrt{2})$
- **C.** because 1 is reserved as the value of $\tan A$ only, never $\sin A$
- **D.** because the triangle is not really a right triangle when both legs are equal

1 more in this set online.

WORKED EXAMPLE · cos A and tan A, using only sin A

1. $\sin^2 A + \cos^2 A = 1$ — *This is the Pythagorean identity, proved earlier.*

2. $\cos A = \sqrt{1 - \sin^2 A}$ — *We rearrange for $\cos A$ and take the positive root, since A is acute.*

3. $\tan A = (\sin A)/(\cos A) = (\sin A)/\sqrt{1-\sin^2 A}$ — *$\tan A$ is $\sin A$ over $\cos A$. We substitute the expression just found for $\cos A$.*

4. for $\sin A = 3/5$, $\cos A = \sqrt{1 - 9/25} = 4/5$ and $\tan A = (3/5)/(4/5) = 3/4$ — CHECK ***Check by squaring $\sin A$ and $\cos A$ and adding them.** $(3/5)^2 + (4/5)^2 = 9/25 + 16/25 = 1$. This confirms the formula for $\cos A$ is right.*

BACK
Any one ratio gives the rest, algebraically · p. 268

TRY
If $\sin A = 3/5$ for an acute angle, what is $\cos A$?

Answer: $\cos A = 4/5$, using the 3, 4, 5 triangle.

RULE
Get $\cos A$ from the identity first, then build $\tan A$ as $\sin A$ over that: never invert the order.

WRITE COSINE AND TANGENT USING ONLY SINE.

1. **Start from the Pythagorean identity** $\sin^2 A + \cos^2 A = 1$ holds for every acute angle.
2. **Solve for cosine** Rearrange to $\cos A = \sqrt{1 - \sin^2 A}$, taking the positive root since A is acute.
3. **Build tangent from sine and cosine** $\tan A$ is $\sin A$ over $\cos A$, so substitute the expression just found for $\cos A$.
4. **Try it with a number** For $\sin A = 3/5$, $\cos A = 4/5$ and $\tan A = 3/4$. Use your own value of $\sin A$ the same way.

CHECK YOURSELF

39. Expressing $\cos A$ using only $\sin A$, for an acute angle A, gives

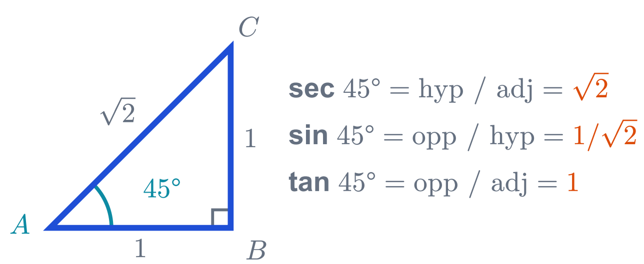
- **A.** $\cos A = 1 - \sin A$
- **B.** $\cos A = \sqrt{1 - \sin^2 A}$
- **C.** $\cos A = \sqrt{1 + \sin^2 A}$
- **D.** $\cos A = 1/(\sin)A$

40. Once $\cos A = \sqrt{1 - \sin^2 A}$ is known, why does $\tan A = (\sin A)/\sqrt{1 - \sin^2 A}$ follow immediately, without any new proof?

- **A.** because $\tan A$ and $\cos A$ happen to be numerically equal for every angle
- **B.** because $\tan A$ is already $(\sin A)/(\cos A)$ by definition
- **C.** because a fresh proof using Pythagoras' theorem is required for $\tan A$ as well
- **D.** because $\tan A$ no longer depends on $\cos A$ once $\sin A$ is known

1 more in this set online.

The check at forty-five degrees



The three ratios the check multiplies are all read off this one triangle: legs 1 and 1, hypotenuse $\sqrt{2}$, and the angle $45°$ at A.

At 45 degrees the two legs match, which is why sine and cosine are equal there and tangent is 1.

WORKED EXAMPLE · Proving sec A (1 - sin A)(sec A + tan A) = 1 without circularity

1. start from the left side alone: $\sec A(1-\sin A)(\sec A+\tan A)$	*A non-circular proof works from one side only. We never touch the right side until the very end. That is the discipline that stops a proof from assuming what it sets out to show.*
2. $=(1/(\cos A))\cdot(1-\sin A)\cdot(1/(\cos A)+(\sin A)/(\cos A))$	*We rewrite* $\sec A$ *and* $\tan A$ *in terms of* $\sin A$ *and* $\cos A$.
3. $=(1/(\cos A))\cdot(1-\sin A)\cdot((1+\sin A)/(\cos A))$	*We combine the two terms inside the bracket, over the common denominator* $\cos A$.
4. $=((1-\sin A)(1+\sin A))/(\cos^2 A)$	*We multiply the three factors together, collecting both* $\cos A$ *terms into* $\cos^2 A$.
5. $=(1-\sin^2 A)/(\cos^2 A)$	$(1-\sin A)(1+\sin A)=1-\sin^2 A$, *a difference of squares.*
6. $=(\cos^2 A)/(\cos^2 A)=1$	***We use*** $\sin^2 A+\cos^2 A=1$ ***once, at the very end, rearranged as*** $1-\sin^2 A=\cos^2 A$***.*** *That is what turns the fraction into* 1*, the right side.*
7. at $A=45°$: $\sec 45°(1-\sin 45°)(\sec 45°+\tan 45°)$ multiplies out to 1	CHECK ***Check the identity at one actual angle.*** *Pick* $A=45°$ *from the standard-angle table. The whole expression collapses to* 1*, exactly as the proof promised for every acute angle.*

BACK
The Pythagorean trigonometric identity · p. 265

TRY
Prove $\operatorname{cosec} A(1-\cos A)(\operatorname{cosec} A+\cot A)=1$, working from the left side only.

Answer: In sin and cos: $((1-\cos A)(1+\cos A))/(\sin^2 A)=(1-\cos^2 A)/(\sin^2 A)=1$.

CAUTION
Reduce the LEFT side alone to 1: cross-multiplying with the right side is the circular-proof trap.

PROVE AN IDENTITY BY REDUCING ONE SIDE ONLY.

1. **Pick one side to work from** Choose the more complicated side of the identity, and leave the other side untouched until the final line.
2. **Rewrite in sine and cosine** Replace every reciprocal or quotient ratio, such as $\sec A$, $\tan A$ or $\cot A$, with its definition in $\sin A$ and $\cos A$.
3. **Combine into one fraction** Put the terms over a common denominator, and multiply out the brackets.
4. **Simplify using the Pythagorean identity** Use $\sin^2 A+\cos^2 A=1$, rearranged if needed, exactly once, to cancel or combine terms.
5. **Confirm it matches the other side** The working should now read exactly as the side left untouched. If it does not, look for an unused simplification.

CHECK YOURSELF

41. To prove $\sec A(1 - \sin A)(\sec A + \tan A) = 1$**, the worked method starts by**

- **A.** assuming the identity is true and checking it numerically for $A = 30°$
- **B.** cross-multiplying both sides of the equation
- **C.** rewriting the left side entirely in terms of $\sin A$ and $\cos A$
- **D.** squaring both sides of the identity first

42. In the proof of $\sec A(1 - \sin A)(\sec A + \tan A) = 1$**, the step** $((1 - \sin A)(1 + \sin A))/(\cos^2 A)$ **simplifies to** $(1 - \sin^2 A)/(\cos^2 A)$**. Which fact justifies this step?**

- **A.** $\sin^2 A + \cos^2 A = 1$, applied at this exact step
- **B.** $\tan A = (\sin A)/(\cos A)$, applied at this exact step
- **C.** the difference-of-squares pattern
- **D.** $\sec A = 1/(\cos A)$, applied at this exact step

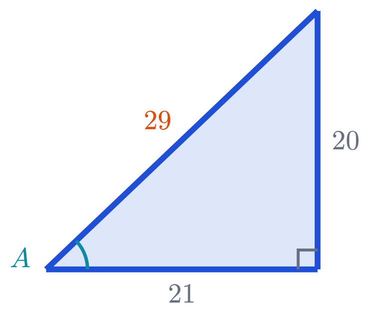
1 more in this set online.

Wrapping up

IN EVERY CHAPTER

Recap

One triangle checks the identity



$\sin A = 20/29$ and $\cos A = 21/29$, and $(20/29)^2 + (21/29)^2 = 1$. One triangle, and the identity checks.

The 20-21-29 triangle checks the identity: sin A = 20/29 and cos A = 21/29, and their squares add to 1.

- SIMILARITY: every ratio depends only on the angle, proved by AA similarity. A 3-4-5 triangle and its 6-8-10 double both give $\sin A = 3/5$.
- STANDARD ANGLES: exact values at $0°$, $30°$, $45°$, $60°$ and $90°$. For instance, $\sin 30° = 1/2$.
- PYTHAGOREAN IDENTITY: $\sin^2 A + \cos^2 A = 1$, checked on the 20-21-29 triangle: $(20/29)^2 + (21/29)^2 = 1$.
- DERIVED IDENTITIES: divide $p^2 + b^2 = h^2$ by a different squared side. Dividing by b^2 gives $1 + \tan^2 A = \sec^2 A$.

- NON-CIRCULAR PROOF: reduce one side alone, never the identity being proved. $\sec A(1 - \sin A)(\sec A + \tan A)$ reduces to 1 this way. No target statement appears in the middle.

Let us step back and see what carried everything so far. Every ratio traces back to one right-triangle picture. Each ratio depends only on the angle, never on the size of the triangle. Similarity proves this; it is not merely asserted.

The standard angles $0°, 30°, 45°, 60°$ and $90°$ give exact values worth knowing outright. $\sin^2 A + \cos^2 A = 1$, and its two derived identities, let any ratio be rewritten. Any one ratio converts into any other this way.

Reducing one side alone, and never assuming the identity being proved, is what makes both proofs trustworthy. The ratios-well-defined proof never assumed the ratios matched before showing it. The Pythagorean identity was built once, from Pythagoras' theorem alone, then reused everywhere else.

Pick any acute angle you like: every identity above still holds for it. You can rebuild any ratio you forget from these two facts alone.

KEY
One triangle picture, one identity (plus its two divisions): everything else follows from these.

BACK
The ratios depend only on the angle · p. 256

RULE
To recall a forgotten ratio, rebuild it from $\sin A$ and $\cos A$, or from a small right triangle, rather than memorising it alone.

CHECK YOURSELF

43. Every ratio in this chapter is built from

- **A.** a separate triangle for each of the six ratios
- **B.** a circle of radius 1, as used for angles beyond 90°
- **C.** measuring many different triangles and averaging the ratios found
- **D.** one right triangle, with ratios fixed by the angle alone

44. This chapter's identities all trace back to one relationship. Which one?

- **A.** Pythagoras' theorem, $p^2 + b^2 = h^2$, divided by different sides' squares
- **B.** the standard-angle table's five exact values
- **C.** the AA similarity criterion from Chapter 6
- **D.** the definitions of $\sin A$, $\cos A$ and $\tan A$ alone, with no further theorem needed

1 more in this set online.

CHECK YOURSELF: THE WHOLE CHAPTER

45. A right triangle has an acute angle A. If the triangle is scaled up so every side becomes exactly twice as long, what happens to $\sin A$?

- **A.** $\sin A$ also doubles, since the side opposite A doubles in length. Doubling the triangle doubles every ratio derived from it, by this reasoning.
- **B.** $\sin A$ becomes half its original value, since the hypotenuse grows faster than the opposite side.
- **C.** $\sin A$ becomes undefined, since a ratio is only meaningful for the triangle it was first measured in.
- **D.** $\sin A$ stays exactly the same, since it depends only on the angle A, not on the size of the triangle.

46. Using the standard-angle table, what is $\cos 60°$?

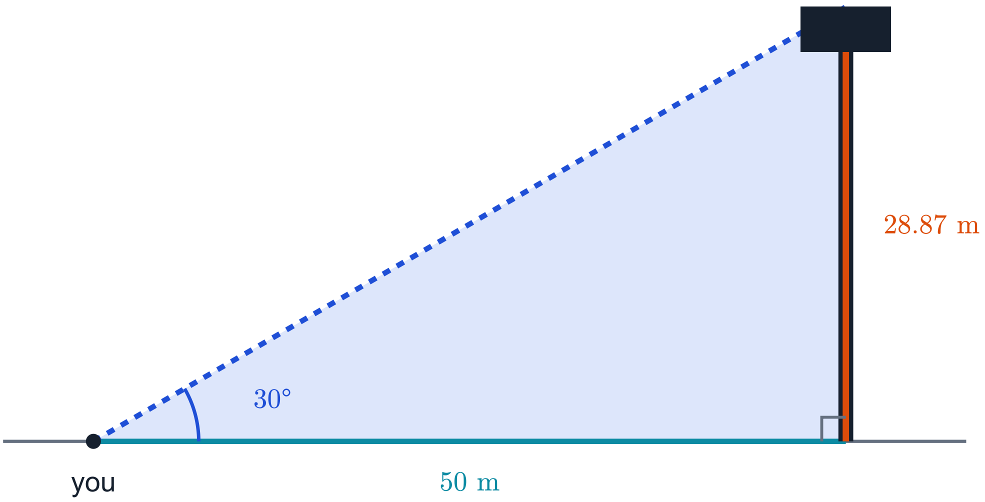
- **A.** $\sqrt{3}/2$
- **B.** $1/\sqrt{2}$
- **C.** $1/2$
- **D.** $\sqrt{3}$

3 more in this set online.

IN EVERY CHAPTER

Where you will meet this

The water tower is fifty tan thirty



$$\text{height} = 50 \cdot \tan 30° = 50/\sqrt{3}$$

One distance along the ground, one angle looked up, and the tangent of that angle turns them into a height nobody had to climb to measure.

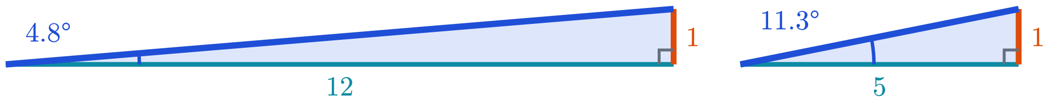
A tower's height comes from a 50 m distance and a 30 degree angle, giving 50 tan 30, about 28.87 m.

You may never draw a right triangle again after Class 10. But sin, cos and tan describe slopes, heights and corners you meet often. Here are seven of those places.

- **Finding a water tower's height.** You stand 50 m from the foot of a water tower. From there, its top is at an angle of elevation of 30°.
  The maths: the height is 50 times $\tan 30°$, read straight off the standard-angle table.
  $\tan 30° = 1/\sqrt{3}$. So the height is $50 \cdot \tan 30° = 50/\sqrt{3} \approx 28.87$ m.
- **Designing a wheelchair ramp.** A shop is building a wheelchair ramp. The rule is 1 m of rise for every 12 m of ground it covers.
  The maths: the tangent of the ramp's angle is $1/12$, since $\tan A$ is the rise over the run.
  That angle is about 4.8°, a gentle ramp.
- **A steep ghat road.** A hill road's sign warns of a slope of 1 in 5. The road rises 1 m for every 5 m of level ground.
  The maths: the road's angle is the one whose tangent is $1/5$. As a slope nears vertical, $\tan$ grows with no limit, unlike $\sin$ and $\cos$, which never pass 1.
  A "1 in 5" slope makes an angle of about 11.3°.

BACK
The three ratios of an acute angle · p. 254

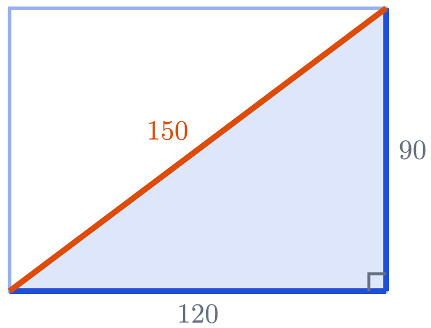
A gentle ramp and a steep road



The tangent of the angle is the rise over the run, so the steeper road has the larger tangent: 1 in 5 against 1 in 12.

Two triangles share a rise of 1 over runs of 12 and 5, giving foot angles of 4.8 and 11.3 degrees.

A diagonal proves the frame square



The corner is square only when the three sides fit Pythagoras' theorem, and 90, 120 and 150 do.

The shelf frame's diagonal: 90 squared plus 120 squared is 22500, and so is 150 squared, so 150 cm proves the corner square.

- **Leaning a ladder to paint a wall.** A 5 m ladder leans against a wall, with its foot 3 m out. It makes an angle A with the ground.
  The maths: the ground, the wall and the ladder make a right triangle. So $\cos A$ is the ground side over the ladder.
  $\cos A = 3/5$. The wall side is $\sqrt{5^2 - 3^2} = 4$ m, so $\sin A = 4/5$ and the ladder reaches 4 m up.
- **Checking a shop shelf frame is square.** You build a shelf frame 90 cm along one side and 120 cm along the other. You want a true right angle at the corner.
  The maths: the corner is square only when the three sides fit Pythagoras' theorem, $a^2 + b^2 = c^2$.
  $90^2 + 120^2 = 8100 + 14400 = 22500$, and $150^2 = 22500$ too. So a 150 cm diagonal proves the frame square.
- **Cutting a diamond kite.** You are cutting a diamond kite. Where the spine meets the cross-stick, one triangle has a 45° corner.
  The maths: since $\tan 45° = 1$, the two straight edges of that triangle are equal.
  If one edge is 21 cm, so is the other, and the slanted edge is $21\sqrt{2} \approx 29.70$ cm.
- **Cutting cloth from a design sheet.** Your shop's banner design sheet gives $\sin A = 3/5$ for one corner of a triangular flag.
  The maths: you get $\cos A$ from $\sin^2 A + \cos^2 A = 1$, without measuring the corner again.
  $\cos A = \sqrt{1 - (3/5)^2} = \sqrt{16/25} = 4/5$.

Your turn. A tailor's pattern for a triangular pocket gives $\cos A = 12/13$ at one corner. What are $\sin A$ and $\tan A$? *Answer:* $\sin A = \sqrt{1-(12/13)^2} = \sqrt{25/169} = 5/13$. So $\tan A = (5/13)/(12/13) = 5/12$.

Practice: Exercise 8.1

PRACTICE SET

1. PRACTICE Given $\tan A = 12/35$ for an acute angle A, find $\sin A$ and $\cos A$. (Worked in full below — read it, then do the next two the same way.)

   SOLUTION — Q1 · **Worked in full**

1.	opposite $= 12k$, adjacent $= 35k$	*one ratio fixes the shape of the triangle but not its size, so carry the size as a factor k and let it cancel later*
2.	hypotenuse $= \sqrt{(12k)^2 + (35k)^2} = \sqrt{1369k^2} = 37k$	*Pythagoras' theorem gives the third side, and the k comes out of the square root as a factor*
3.	$\sin A = (12k)/(37k) = 12/37$ and $\cos A = (35k)/(37k) = 35/37$	*every k cancels, which is why any triangle with this tangent gives the same ratios*
4.	$(12/37)^2 + (35/37)^2 = (144 + 1225)/1369 = 1$	*the check — $\sin^2 A + \cos^2 A$ has to come out at 1, so this catches a slip in the hypotenuse without redoing it*

2. PRACTICE In a right triangle the hypotenuse is 53 cm and one leg is 45 cm. Find $\sin A$ and $\cos A$ for the angle A opposite the other leg, and check your two answers with $\sin^2 A + \cos^2 A = 1$. (Start the same way, but here two sides are given, so Pythagoras' theorem gives the third straight away and there is no k to carry.)
3. PRACTICE Given $\operatorname{cosec} A = 61/11$ for an acute angle A, find $\cos A$ and $\cot A$. (Same first move — cosec is hypotenuse over opposite, so name those two sides first and let Pythagoras' theorem give the third.)
4. PRACTICE Given $\cos A = 5/13$ for an acute angle A, find every other ratio.
5. PRACTICE In a right triangle, the two legs are 8 and 15. Find all six ratios of the angle opposite the shorter leg.
6. PRACTICE Two acute angles P and Q satisfy $\sin P = \sin Q$. What can you conclude about P and Q, and why?
7. PRACTICE Evaluate $(\sin A)/(\cos A) - \tan A$ for any acute angle A.
8. PRACTICE State whether each is true or false, with a reason: (a) $\sin A$ can equal 1.2 for some acute angle A. (b) $\sec A$ is always at least 1.
9. PRACTICE State whether each is true or false, with a reason: (a) $\sin A$ is the product of sin and A. (b) $\tan A$ has no upper bound as A approaches 90°.

ANSWERS **1.** $\sin A = 12/37$ and $\cos A = 35/37$.
2. The other leg is $\sqrt{53^2 - 45^2} = \sqrt{2809 - 2025} = \sqrt{784} = 28$ cm, so $\sin A = 28/53$ and $\cos A = 45/53$. The check gives $(784 + 2025)/2809 = 1$.
3. The adjacent side is $\sqrt{61^2 - 11^2} = \sqrt{3721 - 121} = \sqrt{3600} = 60$, so $\cos A = 60/61$ and $\cot A = 60/11$.
4. $\sin A = 12/13$, $\tan A = 12/5$, $\operatorname{cosec} A = 13/12$, $\sec A = 13/5$, $\cot A = 5/12$.

5. hypotenuse $= \sqrt{8^2+15^2} = 17$; for the angle opposite the leg 8: $\sin A = 8/17$, $\cos A = 15/17$, $\tan A = 8/15$, $\operatorname{cosec} A = 17/8$, $\sec A = 17/15$, $\cot A = 15/8$.
6. $P = Q$ — sin rises without repeating any value as an acute angle runs from 0° to 90°, so equal sines force equal angles.
7. 0 — $\tan A$ is defined as $(\sin A)/(\cos A)$, so the two terms cancel exactly.
8. (a) False — $\sin A$ can never exceed 1. (b) True — $\sec A = 1/(\cos A)$ and $\cos A \le 1$.
9. (a) False — $\sin A$ is one inseparable symbol, not a product. (b) True — $\tan A$ has no upper bound as A approaches 90°.
Full solutions: **manthika.com/learn/math10-ch08**

Further practice

PRACTICE SET

10. PRACTICE Given $\sin A = 7/25$ for an acute angle A, find $\cos A$ and $\tan A$.

SOLUTION — Q10 · **Worked in full**

1.	$x = \sqrt{625-49} = \sqrt{576} = 24$	*$\sin A = 7/25$ means a right triangle with A's opposite side 7 and hypotenuse 25; let x be the adjacent side. Pythagoras' theorem gives $x^2 = 25^2 - 7^2 = 625 - 49$, and x is its positive square root.*
2.	$\cos A = 24/25$	*$\cos A$ is adjacent over hypotenuse, and the adjacent side is $x = 24$.*
3.	$\tan A = 7/24$	*$\tan A$ is opposite over adjacent.*

11. PRACTICE In a right triangle ABC, right-angled at B, $\tan A = 9/40$. Find $\sin A$ and $\cos A$.

12. PRACTICE In a right triangle PQR, right-angled at Q, the legs are PQ $= 7$ cm and QR $= 24$ cm. Find $\sin R$, $\cos R$ and $\tan R$.

13. PRACTICE Given $\cot A = 2/3$ for an acute angle A, find $\sin A$ and $\cos A$.

14. PRACTICE Two acute angles X and Y satisfy $\cos X = \cos Y$. What can you conclude about X and Y, and why?

15. PRACTICE Simplify $\sec A \cdot \sin A \cdot \cot A$ for any acute angle A.

16. PRACTICE State whether true or false, with a reason: (a) $\cot A$ can equal 0 for some acute angle A. (b) $\sin A$ is never negative for an acute angle A.

17. PRACTICE For an acute angle A, $\sin A = 2k$ and $\cos A = k$ for some positive number k. Find k.

ANSWERS 10. $\cos A = 24/25$, $\tan A = 7/24$. 11. $\sin A = 9/41$, $\cos A = 40/41$.
12. $\sin R = 7/25$, $\cos R = 24/25$, $\tan R = 7/24$. 13. $\sin A = 3/\sqrt{13}$, $\cos A = 2/\sqrt{13}$.
14. $X = Y$ — cos falls without repeating any value as an acute angle runs from 0° to 90°, so equal cosines force equal angles.
15. 1.
16. (a) False — $\cot A = (\cos A)/(\sin A)$, and for an acute angle neither $\cos A$ nor $\sin A$ is 0, so $\cot A$ is never 0. (b) True — $\sin A$ is a ratio of positive side lengths, always positive for an acute angle.

17. $k = 1/\sqrt{5}$.
Full solutions: **manthika.com/learn/math10-ch08**

Practice: Exercise 8.2

PRACTICE SET

1. PRACTICE Evaluate $2 \cdot \sin 30° \cdot \cos 30°$. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · Worked in full

1.	$\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$	*read both values off the standard-angle table first, so that substituting is the only thing happening on the next line*
2.	$2 \cdot (1/2) \cdot \left(\sqrt{3}/2\right) = (2 \cdot 1 \cdot \sqrt{3})/(2 \cdot 2) = (2\sqrt{3})/4$	*multiply the tops together and the bottoms together, and leave the cancelling until the whole product is written down*
3.	$(2\sqrt{3})/4 = \sqrt{3}/2$	*cancel the common factor 2*
4.	$2 \cdot \sin 30° \cdot \cos 30° = \cos 30°$	*the check — $\sin 30°$ is exactly $1/2$, so the 2 and the $1/2$ cancel and the whole expression is just $\cos 30°$, which is the answer already found*

2. PRACTICE Evaluate $\sin^2 30° + \cos^2 60° + \tan^2 45°$. (Start the same way — write the three table values down before squaring anything.)

3. PRACTICE In triangle PQR, right-angled at Q, angle $P = 45°$ and PQ $= 7$ cm. Find QR and PR. (The table used the other way round — pick the ratio that joins the side you have to the side you want, then substitute the table value.)

4. PRACTICE Evaluate $\sin 30° \cdot \cos 60° + \cos 30° \cdot \sin 60°$.

5. PRACTICE Evaluate $(\tan 45°)^2 - (\sin 30°)^2$.

6. PRACTICE Choose the correct value of $\sec 60°$: (a) $1/2$ (b) 2 (c) $\sqrt{3}$ (d) $2/\sqrt{3}$.

7. PRACTICE State whether true or false, with a reason: as acute angle A increases from 0° to 90°, $\sin A$ decreases.

8. PRACTICE State whether true or false, with a reason: $\cos 0°$ and $\sin 90°$ are equal.

ANSWERS 1. $\sqrt{3}/2$. 2. $(1/2)^2 + (1/2)^2 + 1^2 = 1/4 + 1/4 + 1 = 3/2$.
3. $\tan 45° = \text{QR}/\text{PQ} = 1$, so QR $= 7$ cm; and $\cos 45° = \text{PQ}/\text{PR} = 1/\sqrt{2}$, so PR $= 7\sqrt{2}$ cm.
4. $(1/2)(1/2) + \left(\sqrt{3}/2\right)\left(\sqrt{3}/2\right) = 1/4 + 3/4 = 1$. 5. $1 - 1/4 = 3/4$. 6. (b) 2.
7. False — $\sin A$ rises from 0 to 1 as A runs from 0° to 90°. 8. True — both equal 1.
Full solutions: **manthika.com/learn/math10-ch08**

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Further practice

PRACTICE SET

9. PRACTICE Evaluate $\sin 60° \cdot \cos 30° - \cos 60° \cdot \sin 30°$.

SOLUTION — Q9 · **Worked in full**

1. $E = \left(\sqrt{3}/2\right)\left(\sqrt{3}/2\right) - (1/2)(1/2)$ — *Substitute the standard values:* $\sin 60° = \sqrt{3}/2$, $\cos 30° = \sqrt{3}/2$, $\cos 60° = 1/2$ *and* $\sin 30° = 1/2$.
2. $E = 3/4 - 1/4 = 1/2$ — *Multiply out and subtract.*

10. PRACTICE Evaluate $\text{cosec}^2 45° - \cot^2 45°$.

11. PRACTICE Evaluate $4\cos^2 45° - 3\sin^2 60°$.

12. PRACTICE Evaluate $(\sin 30° + \cos 60°)/(\sin 45° \cdot \cos 45°)$.

13. PRACTICE Choose the correct value of $\text{cosec } 30°$: (a) 1 (b) 2 (c) $\sqrt{2}$ (d) $2/\sqrt{3}$.

14. PRACTICE State whether true or false, with a reason: (a) $\tan A$ increases as acute angle A increases from 0° to 90°. (b) $\sec 0°$ equals 0.

15. PRACTICE Find $x = 2\sin^2 30° + \text{cosec}^2 60° - 2\cos^2 45°$.

16. PRACTICE In a right triangle ABC, right-angled at B, angle $A = 60°$ and side $BC = 6\sqrt{3}$ cm. Find AB and AC.

ANSWERS **9.** $1/2$. **10.** 1. **11.** $-1/4$. **12.** 2. **13.** (b) 2.
14. (a) True — $\tan A = (\sin A)/(\cos A)$, and $\sin A$ rises while $\cos A$ falls as A runs from 0° to 90°, so the ratio only grows. (b) False — $\sec 0° = 1/(\cos 0°) = 1/1 = 1$, not 0.
15. $x = 5/6$. **16.** AB = 6 cm, AC = 12 cm.
Full solutions: **manthika.com/learn/math10-ch08**

Practice: Exercise 8.3

PRACTICE SET

1. PRACTICE If $\cos A = 12/13$ for an acute angle A, find $\sin A$ and $\tan A$ using the Pythagorean identity, without drawing a triangle. (Worked in full below — read it, then do the next two the same way.)

SOLUTION — Q1 · **Worked in full**

1. $\sin^2 A + \cos^2 A = 1$, so $\sin^2 A = 1 - \cos^2 A$ — *rearrange the identity before a single number goes in, so only one thing happens per line*
2. $\sin^2 A = 1 - (12/13)^2 = 1 - 144/169 = 25/169$ — *substitute and square — the whole of 12/13 is squared, top and bottom*
3. $\sin A = 5/13$ — *take the positive square root, and say why you may — A is acute, so every ratio of it is positive*
4. $\tan A = (\sin A)/(\cos A) = (5/13)/(12/13) = 5/12$ — *the quotient relation finishes it, and the 13s cancel because both ratios came from the same triangle*

SOLUTION — Q1 · Worked in full

5.	$5^2 + 12^2 = 25 + 144 = 169 = 13^2$	*the check — read the identity back on the two answers, which is the same test the first line was built from*

2. PRACTICE Express $\cot A$ in terms of $\sin A$ alone, for an acute angle A. (Start the same way — the identity gives $\cos A$ first, then the quotient relation turns it into $\cot A$.)

3. PRACTICE Prove $(1 - \cos A)(1 + \cos A) = \sin^2 A$, working from the left side only. (Multiply the two brackets out first, then use the Pythagorean identity once. The right side is never touched.)

4. PRACTICE Express $\sec A$ in terms of $\tan A$ alone.

5. PRACTICE Choose the correct option: $1 + \cot^2 A$ equals (a) $\sin^2 A$ (b) $\cos^2 A$ (c) $\operatorname{cosec}^2 A$ (d) $\sec^2 A$.

6. PRACTICE If $\cos A = 3/5$, find $\sin A$ and $\tan A$ using the Pythagorean identity, without drawing a triangle.

7. PRACTICE Prove $(\sin A)/(1 + \cos A) + (1 + \cos A)/(\sin A) = 2 \cdot \operatorname{cosec} A$, working from the left side only.

8. PRACTICE Prove $(1 - \tan^2 A)/(1 + \tan^2 A) = \cos^2 A - \sin^2 A$, working from the left side only.

ANSWERS **1.** $\sin A = 5/13$ and $\tan A = 5/12$.
2. $\cos A = \sqrt{1 - \sin^2 A}$, so $\cot A = (\cos A)/(\sin A) = \sqrt{1 - \sin^2 A}/(\sin A)$.
3. The left side multiplies out to $1 - \cos^2 A$. The identity $\sin^2 A + \cos^2 A = 1$ rearranges to $1 - \cos^2 A = \sin^2 A$, which is the right side. The right side was never altered.
4. $\sec A = \sqrt{1 + \tan^2 A}$, from $1 + \tan^2 A = \sec^2 A$. **5.** (c) $\operatorname{cosec}^2 A$.
6. $\sin A = \sqrt{1 - 9/25} = 4/5$; $\tan A = (4/5)/(3/5) = 4/3$.
7. Combine over the common denominator $\sin A(1 + \cos A)$: the numerator becomes $\sin^2 A + (1 + \cos A)^2 = \sin^2 A + 1 + 2\cos A + \cos^2 A = 2 + 2\cos A$, using the identity. The fraction is $(2(1 + \cos A))/(\sin A(1 + \cos A)) = 2/(\sin A) = 2 \cdot \operatorname{cosec} A$.
8. Multiply numerator and denominator by $\cos^2 A$: $(\cos^2 A - \sin^2 A)/(\cos^2 A + \sin^2 A) = (\cos^2 A - \sin^2 A)/1 = \cos^2 A - \sin^2 A$.
Full solutions: **manthika.com/learn/math10-ch08**

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Further practice

PRACTICE SET

9. PRACTICE If $\sin A = 1/3$ for an acute angle A, find $\cos A$ using the Pythagorean identity.

SOLUTION — Q9 · Worked in full

1.	$\cos^2 A = 1 - (1/3)^2$	*Rearrange* $\sin^2 A + \cos^2 A = 1$ *and substitute* $\sin A = 1/3$.
2.	$\cos^2 A = 1 - 1/9 = 8/9$	*Square and subtract.*

SOLUTION — Q9 · Worked in full

3. $\cos A = \sqrt{8/9} = (2\sqrt{2})/3$ — *Take the positive square root, since A is acute, and simplify $\sqrt{8} = 2\sqrt{2}$.*

10. PRACTICE If $\cos A = 2/3$ for an acute angle A, find $\sin A$ and $\cot A$ using the Pythagorean identity.

11. PRACTICE Given $\sec A = 5/4$ for an acute angle A, find the value of $\tan^2 A$ using the identity $1 + \tan^2 A = \sec^2 A$.

12. PRACTICE Given $\operatorname{cosec} A = 13/12$ for an acute angle A, find the value of $\cot^2 A$ using the identity $1 + \cot^2 A = \operatorname{cosec}^2 A$.

13. PRACTICE Express $\operatorname{cosec} A$ in terms of $\cot A$ alone.

14. PRACTICE State whether true or false, with a reason: (a) $\sin^2 A - \cos^2 A$ can be written as $1 - 2\cos^2 A$. (b) $\sec^2 A - \tan^2 A = 0$ for every acute angle A.

15. PRACTICE Prove $(\cot A - \cos A)/(\cot A + \cos A) = (\operatorname{cosec} A - 1)/(\operatorname{cosec} A + 1)$, working from the left side only.

16. PRACTICE Prove $(\tan A)/(1 - \cot A) + (\cot A)/(1 - \tan A) = 1 + \sec A \cdot \operatorname{cosec} A$, working from the left side only.

ANSWERS **9.** $\cos A = (2\sqrt{2})/3$. **10.** $\sin A = \sqrt{5}/3$, $\cot A = 2/\sqrt{5}$. **11.** $\tan^2 A = 9/16$.
12. $\cot^2 A = 25/144$.
13. $\operatorname{cosec} A = \sqrt{1 + \cot^2 A}$, from $1 + \cot^2 A = \operatorname{cosec}^2 A$, taking the positive root since A is acute.
14. (a) True — since $\sin^2 A = 1 - \cos^2 A$, $\sin^2 A - \cos^2 A = 1 - \cos^2 A - \cos^2 A = 1 - 2\cos^2 A$. (b) False — $\sec^2 A - \tan^2 A = 1$ for every acute angle A, not 0.
15. Write $\cot A = (\cos A)/(\sin A)$. The numerator becomes $(\cos A)/(\sin A) - \cos A = (\cos A \cdot (1 - \sin A))/(\sin A)$ and the denominator becomes $(\cos A)/(\sin A) + \cos A = (\cos A \cdot (1 + \sin A))/(\sin A)$. The fraction is $(1 - \sin A)/(1 + \sin A)$, since $\cos A$ and $\sin A$ cancel. Writing $\operatorname{cosec} A = 1/(\sin A)$ on the right side gives $(1/(\sin A) - 1)/(1/(\sin A) + 1) = (1 - \sin A)/(1 + \sin A)$, the same expression.
16. Write $\tan A = (\sin A)/(\cos A)$ and $\cot A = (\cos A)/(\sin A)$. The left side becomes $(\sin^2 A)/(\cos A \cdot (\sin A - \cos A)) - (\cos^2 A)/(\sin A \cdot (\sin A - \cos A)) = (\sin^3 A - \cos^3 A)/(\sin A \cdot \cos A \cdot (\sin A - \cos A))$. Since $\sin^3 A - \cos^3 A = (\sin A - \cos A) \cdot (\sin^2 A + \sin A \cdot \cos A + \cos^2 A) = (\sin A - \cos A) \cdot (1 + \sin A \cdot \cos A)$, the $(\sin A - \cos A)$ factor cancels, leaving $(1 + \sin A \cdot \cos A)/(\sin A \cdot \cos A) = 1/(\sin A \cdot \cos A) + 1 = \sec A \cdot \operatorname{cosec} A + 1$, the right side.
Full solutions: **manthika.com/learn/math10-ch08**